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Numerical properties of generalized discrepancies on spheres of arbitrary dimension ☆

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ABSTRACT

Quantifying uniformity of a configuration of points on the sphere is an interesting topic that is receiving growing attention in numerical analysis. An elegant solution has been provided by Cui and Freeden [J. Cui, W. Freeden, Equidistribution on the sphere, SIAM J. Sci. Comput. 18 (2) (1997) 595–609], where a class of discrepancies, called *generalized discrepancies* and originally associated with pseudodifferential operators on the unit sphere in $\mathbb{R}^3$, has been introduced. The objective of this paper is to extend to the sphere of arbitrary dimension this class of discrepancies and to study their numerical properties. First we show that generalized discrepancies are diaphonies on the hypersphere. This allows us to completely characterize the sequences of points for which convergence to zero of these discrepancies takes place. Then we discuss the worst-case error of quadrature rules and we derive a result on tractability of multivariate integration on the hypersphere. At last we provide several versions of Koksma–Hlawka type inequalities for integration of functions defined on the sphere.

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1. Introduction

Quantifying uniformity of a configuration of points on a sphere is an interesting topic that is receiving growing attention in numerical analysis (see, e.g., [27,21,22,39,40,12,46,30,32,2,20,3,13, 48,24,8,34] and, for an earlier example, [50]; the problem has been dealt with extensively also in statistics, see, e.g., [51,4,19,35,37,38]). An elegant solution has been provided in [12], where a class of discrepancies, called *generalized discrepancies* and originally associated to pseudodifferential operators on the sphere in $\mathbb{R}^3$, has been introduced.[1]

The objective of this paper is to extend to the sphere of arbitrary dimension this class of discrepancies and to study their numerical properties. We first introduce the generalized discrepancy of a configuration of points on the sphere $\mathbb{S}^r$ (by which we denote the unit sphere in $\mathbb{R}^{r+1}$). In the original paper [12], generalized discrepancies were defined using pseudodifferential operators (see Eq. (3) below for a definition). However, our definition involves only sequences of weights that can sometimes be made to coincide with the symbol of a pseudodifferential operator.

First, we show that generalized discrepancies are diaphonies in the sense of [1] defined in the setting of a certain reproducing kernel Hilbert space H over the hypersphere. This allows us to completely characterize the sequences of points for which convergence to 0 of generalized discrepancies takes place, thus answering a question raised in [18]. Moreover, we show that the generalized discrepancy associated with the above-defined Hilbert space H coincides with the worst-case error for an equal-weight numerical integration rule applied to a function $f \in \mathsf{B}(H)$, where $\mathsf{B}(H)$ is the unit ball in H (see [48, Section 4], for the case $r = 2$ and a particular choice of the symbol, and [9, Section 2.2], for a treatment with different aims of the worst-case error in the special case of Sobolev spaces). Using this representation, we provide a tractability result for numerical integration by means of quadrature rules with nonnegative weights on the hypersphere. This appears to be the first result of this type in the literature. Then we show that the above derived conditions are necessary and sufficient, under an additional requirement, for tractability and strong tractability.

Then we provide several versions of Koksma–Hlawka type inequalities for integration of functions defined on the sphere. First of all, we state an inequality for functions belonging to the space H (Proposition 12) and we use it to generalize to the sphere of arbitrary dimension (Corollary 13) the inequality stated in [12] for functions belonging to a Sobolev class. Then we show that generalized discrepancies appear also in three versions of the Koksma–Hlawka inequality, the first one for polynomials (Proposition 15), the second one for functions satisfying a Lipschitz-type condition (Proposition 17) and the third one for indicator functions of K-regular sets (Proposition 19). These inequalities can be linked to some Erdős–Turán inequalities on the sphere that have appeared in the literature (see [27,21,32]).

The topic of uniformity of configurations of points on the sphere $\mathbb{S}^r$ has received a lot of attention also in statistics, where it falls under the names of circular, spherical and directional statistics. When applied to a sample of random points on the sphere, generalized discrepancies provide a flexible class of statistical tests of uniformity encompassing the commonly used Ajne, Beran, Bingham, Giné, Hermans–Rasson, Jammalamadaka, Prentice, Pycke, Rayleigh, Rothman and Watson test statistics (see below for details and references). The asymptotic statistical properties of these discrepancies will be derived in a companion paper.

We generalize several results from the literature. Corollary 13 generalizes Theorem 3.1 in [12] to higher-dimensional spheres. Theorems 1 and 2 in [13] can be obtained as Corollaries of our results. The connection between generalized discrepancies and the worst-case error for an equal-weight numerical integration rule gives rise to higher-dimensional analogues of the results in Section 4 of [48] and of their Theorem 4.1, and to a generalized version of the results in Section 3.3 in [7]. In the first reference the authors deal with the case $r = 2$ when the symbol is given by the choice in [12, p. 598], while in the second the author deals with the case in which the elements of the sequence of weights

[1] The same name was used by Hickernell [25] to denote a different class of discrepancies on the unit hypercube, whose statistical properties were studied in [11].

are not allowed to take infinite values; we extend these results to the case of general sequences of weights with $r \geq 1$.

Moreover, we fill several gaps in the literature. Proposition 6 answers a question raised in [18] about sequences of points for which generalized discrepancies and the integration error on the sphere converge to 0. Theorem 10 seems to be the first result in the literature on tractability of integration on the hypersphere by means of quadrature rules with nonnegative weights (see [31] though for the related case of the product of spheres). Proposition 19 provides a rigorous version of Eq. (13) in [12] for approximation of indicator functions on the sphere.

The mathematical tools that are necessary to develop the properties of generalized discrepancies are presented in Section 2. Section 3 collects the relation between generalized discrepancies and the worst-case error for an equal-weight numerical integration rule, the tractability result for integration and the Koksma–Hlawka inequalities. Proofs are gathered in Section 4.

2. Mathematical preliminaries

The following presentation draws heavily on the treatment in [24].[2] We denote by $\mathbb{S}^r$ the sphere in $\mathbb{R}^{r+1}$, by $\omega_r(\cdot)$ the non-normalized surface area on $\mathbb{S}^r$, by $\omega_r \triangleq \omega_r(\mathbb{S}^r) = \frac{2\pi^{\frac{r+1}{2}}}{\Gamma\left(\frac{r+1}{2}\right)}$ the surface area of the unit sphere and by $\omega_r^\star(\cdot) \triangleq \frac{\omega_r(\cdot)}{\omega_r}$ the normalized surface area on $\mathbb{S}^r$. Let $L_2(\mathbb{S}^r)$ be the Hilbert space of square-integrable functions on the sphere $\mathbb{S}^r \subset \mathbb{R}^{r+1}$ endowed with the inner product

$$\langle f, g\rangle_{L_2(\mathbb{S}^r)} \triangleq \int_{\mathbb{S}^r} f(\mathbf{x})\, g(\mathbf{x})\, \mathrm{d}\omega_r^\star(\mathbf{x})$$

and the induced norm

$$\|f\|_{L_2(\mathbb{S}^r)} \triangleq \sqrt{\int_{\mathbb{S}^r} [f(\mathbf{x})]^2\, \mathrm{d}\omega_r^\star(\mathbf{x})}.$$

The space of continuous functions on the sphere $\mathbb{S}^r$ is denoted by $\mathcal{C}(\mathbb{S}^r)$, and is endowed with the supremum norm $\|f\|_{\mathcal{C}(\mathbb{S}^r)} \triangleq \sup_{\mathbf{x}\in\mathbb{S}^r} |f(\mathbf{x})|$. The space of functions with continuous derivatives of order k on $\mathbb{S}^r$ is denoted by $\mathcal{C}^k(\mathbb{S}^r)$, while $\mathcal{C}^\infty(\mathbb{S}^r)$ is the space of infinitely often differentiable functions on $\mathbb{S}^r$. In the following, $\mathbb{N}_0$ and $\mathbb{N}$ respectively indicate the set of natural numbers with and without the 0 element.

The restriction of a harmonic homogeneous polynomial in $r+1$ real variables of exact degree ℓ to the sphere $\mathbb{S}^r$ is called a *spherical harmonic of degree* ℓ. The space of all spherical harmonics on $\mathbb{S}^r$ of degree $\ell \in \mathbb{N}_0$ is denoted by $\mathcal{H}_\ell(\mathbb{S}^r)$. The dimension of $\mathcal{H}_\ell(\mathbb{S}^r)$ is given by $\dim[\mathcal{H}_\ell(\mathbb{S}^r)] = N(r-1,\ell)$, where

$$N(r-1,\ell) = \frac{(2\ell+r-1)(\ell+r-2)!}{(r-1)!\ell!}$$

for $\ell \in \mathbb{N}_0$. This implies by the way that $N(r-1,0) = 1$. Any two spherical harmonics of different degree are orthogonal to each other. We will denote as $\left\{Y_{\ell k}^{(r)} | k = 1,\ldots,N(r-1,\ell)\right\}$ an $L_2(\mathbb{S}^r)$-orthonormal basis for $\mathcal{H}_\ell(\mathbb{S}^r)$ of real-valued spherical harmonics of degree ℓ. In particular, we use the conventions $Y_{01}^{(r)}(\mathbf{x}) \equiv 1$ and

$$\int_{\mathbb{S}^r} \left|Y_{\ell k}^{(r)}(\mathbf{x})\right|^2 \mathrm{d}\omega_r^\star(\mathbf{x}) = 1$$

for $\ell \geq 1$ and $k = 1,\ldots,N(r-1,\ell)$.

2 For the interested reader, a classical reference on the topic is [33].

The spherical harmonics of degree ℓ satisfy the so-called addition theorem, stating that the following equality holds:

$$\sum_{k=1}^{N(r-1,\ell)} Y_{\ell k}^{(r)}(\mathbf{x})\, Y_{\ell k}^{(r)}(\mathbf{y}) = N(r-1,\ell)\cdot P_{\ell}^{\frac{r-1}{2}}(\mathbf{x}\cdot\mathbf{y}), \quad \mathbf{x},\mathbf{y}\in\mathbb{S}^r.$$

Here $P_\ell^\lambda(\cdot)$ is the *normalized Gegenbauer* (sometimes also called *ultraspherical*) *polynomial of index* λ *and degree* ℓ, given by

$$P_\ell^\lambda(t) \triangleq \frac{C_\ell^\lambda(t)}{C_\ell^\lambda(1)}, \quad t\in[-1,+1],$$

where the *Gegenbauer* or *ultraspherical polynomials* C_ℓ^λ *of index* λ *and degree* ℓ are defined as orthogonal polynomials on the interval $[-1,+1]$ with respect to the weight function $\left(1-x^2\right)^{\lambda-\frac{1}{2}}$ (see, e.g., [28, Section 4.5]). We recall that $C_\ell^\lambda(1) = \frac{(2\lambda)_\ell}{\ell!}$, where $(x)_n = \Gamma(x+n)/\Gamma(x)$ denotes the Pochhammer symbol, and that $\max_{t\in[-1,1]}\left|C_\ell^\lambda(t)\right| = C_\ell^\lambda(1)$, so that $\max_{t\in[-1,1]}\left|P_\ell^\lambda(t)\right| = 1$. Notice that this implies that $\sum_{k=1}^{N(r-1,\ell)}\left|Y_{\ell k}^{(r)}(\mathbf{x})\right|^2 = N(r-1,\ell)$ and $\left|Y_{\ell k}^{(r)}(\mathbf{x})\right| \le \sqrt{N(r-1,\ell)}$.

The space of spherical polynomials on $\mathbb{S}^r$ of degree at most n is denoted by $\mathcal{P}_n(\mathbb{S}^r)$ and it can be shown that

$$\mathcal{P}_n\left(\mathbb{S}^r\right) = \bigoplus_{\ell=0}^{n}\mathcal{H}_\ell\left(\mathbb{S}^r\right).$$

Therefore, $\dim\left[\mathcal{P}_n(\mathbb{S}^r)\right] = \sum_{\ell=0}^{n} N(r-1,\ell) = N(r,n)$. The union of the sets $\{Y_{\ell k}^{(r)} | k = 1,\ldots,N(r-1,\ell)\}$ for every $\ell\in\mathbb{N}_0$ is a complete orthonormal system in $L_2(\mathbb{S}^r)$. This implies that every function $f\in L_2(\mathbb{S}^r)$ can be expanded into a Fourier series with respect to this orthonormal system, so that the following equality holds in $L_2(\mathbb{S}^r)$ (see also [5]):

$$f = \sum_{\ell=0}^{\infty}\sum_{k=1}^{N(r-1,\ell)} \widehat{f}_{\ell k}^{(r)}\cdot Y_{\ell k}^{(r)}, \tag{1}$$

where the Fourier coefficients are given by

$$\widehat{f}_{\ell k}^{(r)} \triangleq \int_{\mathbb{S}^r} f(\mathbf{x})\, Y_{\ell k}^{(r)}(\mathbf{x})\, \mathrm{d}\omega_r^\star(\mathbf{x}).$$

A result that will turn out to be useful in the following is the fact that f and a centered version of f are linked by the equality

$$f - \int_{\mathbb{S}^r} f(\mathbf{x})\,\mathrm{d}\omega_r^\star(\mathbf{x}) = \sum_{\ell=1}^{\infty}\sum_{k=1}^{N(r-1,\ell)}\widehat{f}_{\ell k}^{(r)}\cdot Y_{\ell k}^{(r)}. \tag{2}$$

The Sobolev space $H^s(\mathbb{S}^r)$, for $s\ge 0$, is the completion of $\bigoplus_{\ell=0}^{\infty}\mathcal{H}_\ell(\mathbb{S}^r)$ with respect to the norm

$$\|f\|_{s,r} \triangleq \sqrt{\sum_{\ell=0}^{\infty}\left(\ell+\frac{r-1}{2}\right)^{2s}\cdot\sum_{k=1}^{N(r-1,\ell)}\left|\widehat{f}_{\ell k}^{(r)}\right|^2}.$$

The space $H^s(\mathbb{S}^r)$ is a Hilbert space when endowed with the inner product

$$\langle f,g\rangle_{s,r} \triangleq \sum_{\ell=0}^{\infty}\left(\ell+\frac{r-1}{2}\right)^{2s}\cdot\sum_{k=1}^{N(r-1,\ell)}\widehat{f}_{\ell k}^{(r)}\widehat{g}_{\ell k}^{(r)}, \quad f,g\in H^s\left(\mathbb{S}^r\right).$$

Clearly, $H^0(\mathbb{S}^r) = L_2(\mathbb{S}^r)$ and $H^t(\mathbb{S}^r)\subset H^s(\mathbb{S}^r)$ for $t\ge s$. The following further Embedding Theorem generalizes Lemma 2.1 in [12]. The proof is a consequence of [24, p. 420] and more generally of Sobolev's Lemma (see [52, pp. 174–175]).

Proposition 1. *$H^s(\mathbb{S}^r) \subset \mathcal{C}^k(\mathbb{S}^r)$ for $s > \frac{r}{2} + k$.*

Let $\{A_\ell\}_{\ell \in \mathbb{N}_0}$ be a sequence of real numbers satisfying

$$\lim_{\ell \to \infty} \frac{|A_\ell|}{\left(\ell + \frac{r-1}{2}\right)^t} = \text{const} \neq 0$$

for a certain $t \in \mathbb{R}$. Then a *pseudodifferential operator of order t*, say $\mathbf{A} : H^s(\mathbb{S}^r) \to H^{s-t}(\mathbb{S}^r)$, is an operator defined by

$$\mathbf{A}f = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(r-1,\ell)} A_\ell \widehat{f}_{\ell k}^{(r)} \cdot Y_{\ell k}^{(r)}, \quad f \in H^s\left(\mathbb{S}^r\right). \tag{3}$$

The sequence $\{A_\ell\}_{\ell \in \mathbb{N}_0}$ is called the *spherical symbol* of $\mathbf{A}$. Several examples and properties of pseudodifferential operators are listed in [12, pp. 597–598]. It is of interest to recall here that, when $\mathbf{A}$ is a pseudodifferential operator of order s, the Sobolev space $H^s(\mathbb{S}^r)$ can be equivalently expressed as

$$H^s\left(\mathbb{S}^r\right) = \left\{f : \mathbb{S}^r \to \mathbb{R} | \mathbf{A}f \in L_2\left(\mathbb{S}^r\right)\right\}.$$

3. Numerical properties

We give the following definition of *generalized discrepancy*, that will turn out to be relevant in the following.

Definition 2. Let $\{A_\ell\}_{\ell \in \mathbb{N}}$ be a sequence of weights such that $A_\ell \in \overline{\mathbb{R}}, A_\ell \neq 0$ for $\ell \geq 1$, and $\sum_{\ell=1}^{\infty} \frac{N(r-1,\ell)}{A_\ell^2} < +\infty$. The generalized discrepancy of a configuration of points $\mathbf{P}_N = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\} \subset \mathbb{S}^r$ associated with $\{A_\ell\}_{\ell \in \mathbb{N}}$ is defined by

$$D\left(\mathbf{P}_N; \{A_\ell\}\right) = \left[\sum_{\ell=1}^{\infty} \frac{1}{A_\ell^2} \cdot \sum_{k=1}^{N(r-1,\ell)} \left| \frac{1}{N} \cdot \sum_{i=1}^{N} Y_{\ell k}^{(r)}\left(\mathbf{x}_i\right) \right|^2 \right]^{\frac{1}{2}}.$$

The same formula applies also when $\{A_\ell\}_{\ell \in \mathbb{N}_0}$ is the symbol of a pseudodifferential operator $\mathbf{A}$ of order s, where $s > \frac{r}{2}$.

Remark 3. Using the addition theorem for spherical harmonics, it turns out that an alternative way of writing $D\left(\mathbf{P}_N; \{A_\ell\}\right)$ is the following one:

$$D\left(\mathbf{P}_N; \{A_\ell\}\right) = \frac{1}{N} \left[\sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{\ell=1}^{\infty} \frac{N(r-1,\ell)}{A_\ell^2} \cdot P_\ell^{\frac{r-1}{2}}\left(\mathbf{x}_i \cdot \mathbf{x}_j\right)\right]^{1/2}.$$

Analogous properties to the ones in Lemma 3.1 in [12] apply in this case too.

The previous Definition 2 encompasses several tests of uniformity proposed in the statistical literature. The link between discrepancies and test statistics is well-established (see, e.g., [26]), since both aim at measuring uniformity by means of the discrepancy between a probability measure and a finitely supported approximation of it. Nevertheless, the general tools required by numerical analysis and statistics are different and this has limited the exchange of results between the two fields. In particular, the instances of statistical tests of uniformity also appearing in numerical analysis as figures-of-merit are limited to the most classical ones. Now we briefly recall the more commonly used test statistics for evaluating uniformity on the circle $\mathbb{S}^1$, the usual sphere $\mathbb{S}^2$ and the hyperspheres in Euclidean spaces of arbitrary order and we show that they fit in the previous framework. The statistic considered by Prentice ([35, p. 170], see also [36]) and inspired by the tests of Giné [19] on $\mathbb{S}^2$ is given by (we changed the normalization from N^{-1} in the original source to N^{-2})

$$T_{r,N}\left(\{a_\ell\}\right) = \frac{1}{N^2} \cdot \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{\ell=1}^{\infty} N\left(r-1,\ell\right) \cdot a_\ell^2 \cdot P_\ell^{\frac{r-1}{2}}\left(\mathbf{x}_i \cdot \mathbf{x}_j\right),$$

where

$$a_\ell = \frac{\omega_{r-1}}{\omega_r} \cdot \left[\int_{-1}^{+1} f(z) P_\ell^{\frac{r-1}{2}}(z) \left(1-z^2\right)^{\frac{r}{2}-1} \mathrm{d}z\right]$$

and $f : \mathbb{R} \to \mathbb{R}$. The identification between $T_{r,N}\left(\{a_\ell\}\right)$ and $D^2\left(\mathbf{P}_N; \{A_\ell\}\right)$ can be done taking $A_\ell^2 = \frac{1}{a_\ell^2}$. In particular, taking $A_1^2 = 1$ and $A_\ell^2 = +\infty$ for any $\ell \geq 1$, we get the $\mathbb{S}^r$ generalization of Rayleigh's test statistic. If $A_2^2 = 1$ and $A_\ell^2 = +\infty$ for any $\ell \neq 2$, we get the $\mathbb{S}^r$ generalization of Bingham's test statistic. Taking

$$A_{2\ell-1}^2 = \left[\frac{\omega_r}{\omega_{r-1}} \cdot \frac{(r/2)_\ell}{(-1/2)_\ell}\right]^2$$

and $A_{2\ell}^2 = +\infty$, we get the $\mathbb{S}^r$ generalization of Ajne's test statistic, often called $A_{r,N}$; the test coincides with the one of Watson [51] on $\mathbb{S}^1$ and of Beran [4] on $\mathbb{S}^2$. Taking

$$A_{2\ell}^2 = \frac{2\ell + r}{r\left(\ell - 1/2\right)} \cdot \left[\frac{(r/2)_\ell}{(-1/2)_\ell}\right]^2$$

and $A_{2\ell+1}^2 = +\infty$, we get Giné's [19, p. 1262] test statistic, often called $G_{r,N}$. Also tests obtained as weighted sums of $A_{r,N}$ and $G_{r,N}$ are sometimes used: the new coefficients $\left\{A_\ell^2\right\}_{\ell \in \mathbb{N}}$ can be obtained from the previous ones. Other statistics on $\mathbb{S}^2$ that can be embedded in this framework have been proposed in [41,43,23,37,38], among others. It is interesting to remark that some of these statistics have $A_\ell^{-1} = 0$ for some values of ℓ. The motivation may be to avoid the computational complexity introduced by an extended summation over the ℓ index or to provide tests that perform well in special instances, such as for example on the real projective space (see the discussion in Section 3.1). Whatever the reason, keeping in mind the potential applications of our results to the above test statistics, in the following we will be particularly careful in allowing for $A_\ell^{-1} = 0$ for some values of ℓ.

3.1. *Generalized discrepancies as diaphonies*

In the following we show that generalized discrepancies can be written as diaphonies in the sense of [1]. This is based on the identification of a suitable reproducing kernel Hilbert space of functions, say H, such that the discrepancy is the Hilbert norm of a well-chosen function.

The elements of the reproducing kernel Hilbert space H are given by weighted sums of spherical harmonics, whose weights respect some summability conditions. One of the features of H is that the contribution of some of the spaces $\mathcal{H}_\ell\left(\mathbb{S}^r\right)$ (the space of spherical harmonics of degree ℓ, coinciding with an eigenspace of the Laplacian on the sphere) can be null. This has an impact on the formula of the discrepancy, since the eigenfunctions belonging to these spaces $\mathcal{H}_\ell\left(\mathbb{S}^r\right)$ do not appear in $D\left(\mathbf{P}_N; \{A_\ell\}\right)$. This generality is needed both in numerical analysis and in statistics. In numerical analysis, it provides a simple way to simplify the class of functions under scrutiny by removing the sets of spherical harmonics corresponding to certain degrees. As an example, if we remove all the spaces $\mathcal{H}_\ell\left(\mathbb{S}^r\right)$ with $\ell > n$, then H will be the space of polynomials of degree up to n defined on the sphere $\mathbb{S}^r$. Another particularly interesting case concerns the (real) projective space $\mathbb{RP}^r$. Since the eigenspace E_k of the Laplacian on $\mathbb{RP}^r$ can be identified with the eigenspace $E_{2k} = \mathcal{H}_{2k}\left(\mathbb{S}^r\right)$ of the Laplacian on $\mathbb{S}^r$, any function on $\mathbb{RP}^r$ can be described by removing all the spaces $\mathcal{H}_\ell\left(\mathbb{S}^r\right)$ with odd ℓ. Therefore, if numerical integration on $\mathbb{RP}^r$ is under scrutiny, we set $A_\ell = +\infty$ for odd ℓ. In statistics, on the other hand, many well-known statistics for uniformity are obtained setting $A_\ell = +\infty$ for some $\ell \in \mathbb{N}$ in the Definition 2 (see the discussion after Remark 3 for some examples).

As a by-product, we provide two results. The first one concerns the set of sequences for which $D^2(\mathbf{P}_N; \{A_\ell\}) \to 0$: in particular we give a characterization of sequences of points $\mathbf{P}_N$ that are 1 -uniformly distributed and uniformly distributed w.r.t. $\omega_r^\star(\cdot)$ (see below for the definitions, taken from [1], respectively on p. 501 and p. 506). The second result allows us to link a generalized discrepancy to the worst-case error for an equal-weight numerical integration rule applied to a function $f \in \mathsf{B}(H)$, where $\mathsf{B}(H)$ is the unit ball in the Hilbert space H; our exposition generalizes the derivation for the case $r = 2$ with a special choice of the sequence of weights that is dealt with in [48, Section 4].

Theorem 4. *The generalized discrepancy $D(\mathbf{P}_N; \{A_\ell\})$ coincides with the diaphony in the sense of [1, p. 501] associated with the reproducing kernel Hilbert space*[3]

$$H = \left\{ f : \mathbb{S}^r \to \mathbb{R} \left| \sum_{\ell=0}^{+\infty} \sum_{k=1}^{N(r-1,\ell)} A_\ell^2 \left| \widehat{f}_{\ell k}^{(r)} \right|^2 < +\infty, \widehat{f}_{\ell k}^{(r)} = 0 \text{ if } A_\ell^2 = +\infty \right. \right\}$$

with inner product

$$(f, g)_H = \sum_{\ell=0}^{+\infty} \sum_{k=1}^{N(r-1,\ell)} A_\ell^2 \widehat{f}_{\ell k}^{(r)} \widehat{g}_{\ell k}^{(r)},$$

norm $\|f\|_H = (f, f)_H^{1/2}$ and kernel

$$K(\mathbf{x}, \mathbf{y}) = \sum_{\ell=0}^{+\infty} \sum_{k=1}^{N(r-1,\ell)} \frac{1}{A_\ell^2} \cdot Y_{\ell k}^{(r)}(\mathbf{x}) Y_{\ell k}^{(r)}(\mathbf{y}) = \sum_{\ell=0}^{+\infty} \frac{N(r-1,\ell)}{A_\ell^2} \cdot P_\ell^{\frac{r-1}{2}}(\mathbf{x} \cdot \mathbf{y}).$$

The previous formulas hold with the understanding that $A_0 = 1$.

Remark 5. Notice that if $c_1 \ell^{\alpha/2} \le A_\ell \le c_2 \ell^{\alpha/2}$ for $\ell \ge 1$ and $\alpha > r$, with (c_1, c_2) a pair of strictly positive real constants, then the norm $\|f\|_H$ is equivalent to the $H^{\frac{\alpha}{2}}(\mathbb{S}^r)$ norm defined above.

3.2. Uniformity properties of generalized discrepancies

A fundamental property of generalized discrepancies that is not shown in [12] is convergence to 0 of this quantity for uniformly distributed sequences. This point is raised in [18, p. 318], where it is stated that this has been proved only for spherical t-designs. Following the approach of [1, p. 501, Definition 1], we use the previously identified space H (see Theorem 4) in order to obtain results about sequences for which convergence to 0 of $D(\mathbf{P}_N; \{A_\ell\})$ holds. We indicate with $\{\mathbf{x}_i\}$ an infinite sequence of points, and with $\mathbf{P}_N$ the first N elements of this sequence.

We say that a sequence of points $\{\mathbf{x}_i\}$ on $\mathbb{S}^r$ is g-uniformly distributed if, for every function $f \in H$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} f(\mathbf{x}_i) = (f, g)_H.$$

In particular, we will be interested in 1-uniform distribution, namely in the fact that

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} f(\mathbf{x}_i) &= (f, 1)_H = A_0^2 \widehat{f}_{01}^{(r)} \widehat{1}_{01}^{(r)} \\ &= \int_{\mathbb{S}^r} f(\mathbf{x}) \, \mathrm{d}\omega_r^\star(\mathbf{x}) \end{aligned} \tag{4}$$

[3] Both the space and the inner product are defined with the understanding that $\infty \cdot 0 = 0$.

for every $f \in H$. We will also need the concept of uniform distribution of a sequence of points with respect to a probability measure (in our case $\omega_r^\star(\cdot)$; see [1, p. 506]). We say that a sequence of points $\{\mathbf{x}_i\}$ on $\mathbb{S}^r$ is uniformly distributed w.r.t. $\omega_r^\star(\cdot)$ if, for every function $f \in \mathcal{C}(\mathbb{S}^r)$,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} f(\mathbf{x}_i) = \int_{\mathbb{S}^r} f(\mathbf{x})\, \mathrm{d}\omega_r^\star(\mathbf{x}). \tag{5}$$

Notice that in our case the difference between these two concepts is the class of functions for which convergence takes place (H in the first case, $\mathcal{C}(\mathbb{S}^r)$ in the second case).

Proposition 6. *The following two facts hold true:*

(i) $\lim_{N\to\infty} D(\mathbf{P}_N; \{A_\ell\}) = 0$ *iff the sequence of points* $\{\mathbf{x}_i\}$ *is* 1*-uniformly distributed for functions* $f \in H$*, where* H *is the Hilbert space of Theorem 4.*

(ii) *If* $A_\ell < +\infty$ *for any* $\ell \in \mathbb{N}$*, then* $\lim_{N\to\infty} D(\mathbf{P}_N; \{A_\ell\}) = 0$ *iff the sequence of points* $\{\mathbf{x}_i\}$ *is uniformly distributed w.r.t.* $\omega_r^\star(\cdot)$*. On the other hand, the fact that* $\lim_{N\to\infty} D(\mathbf{P}_N; \{A_\ell\}) = 0$ *for any sequence of points* $\{\mathbf{x}_i\}$ *that is uniformly distributed w.r.t.* $\omega_r^\star(\cdot)$ *and only for such sequences implies that* $A_\ell < +\infty$ *for any* $\ell \in \mathbb{N}$*.*

Remark 7. Point (ii) of the proposition shows the equivalence between finiteness of A_ℓ for any ℓ and convergence to 0 of the discrepancy for and only for sequences uniformly distributed w.r.t. $\omega_r^\star(\cdot)$. Indeed, when $A_\ell < +\infty$ for any ℓ the two concepts of uniformity coincide: this is the case of the discrepancy considered in [12, p. 602]. On the other hand, when $A_\ell = +\infty$ for some ℓ, 1-uniform distribution reduces the set of functions over which convergence has to take place: this widens the set of sequences for which the result holds. The difference between the two concepts of uniform distribution can be shown through the following discrepancies (known respectively in statistics as Rayleigh's and Bingham's statistics and originally introduced to evaluate uniformity on $\mathbb{S}^2$). Consider the generalization of Rayleigh's statistic to $\mathbb{S}^r$:

$$\begin{aligned} D^2(\mathbf{P}_N; \{A_\ell\}) &= \frac{(r+1)}{N^2} \cdot \sum_{i=1}^{N}\sum_{j=1}^{N} P_1^{\frac{r-1}{2}}\left(\mathbf{x}_i \cdot \mathbf{x}_j\right) \\ &= \sum_{k=1}^{r+1}\left[\frac{1}{N}\cdot\sum_{i=1}^{N} Y_{1k}^{(r)}(\mathbf{x}_i)\right]^2 \end{aligned}$$

that can be obtained by taking $A_1^2 = 1$ and $A_i^2 = +\infty$ for $i > 1$. Rayleigh's statistic converges to 0 for any sequence of points that is 1-uniformly distributed for functions $f \in H$, where

$$\begin{aligned} H &= \left\{ f : \mathbb{S}^r \to \mathbb{R} \,\middle|\, \left|\widehat{f}_{01}^{(r)}\right|^2 + \sum_{k=1}^{r+1}\left|\widehat{f}_{1k}^{(r)}\right|^2 < +\infty, \widehat{f}_{\ell k}^{(r)} = 0 \text{ for } \ell > 1 \right\} \\ &= \mathcal{H}_0\left(\mathbb{S}^r\right) \oplus \mathcal{H}_1\left(\mathbb{S}^r\right) = \mathcal{P}_1\left(\mathbb{S}^r\right). \end{aligned}$$

Therefore a sequence of points for which Rayleigh's statistic converges to 0 is 1-uniformly distributed and integrates all functions belonging to $\mathcal{P}_1(\mathbb{S}^r)$. The same reasoning applies to Bingham's statistic:

$$\begin{aligned} D^2(\mathbf{P}_N; \{A_\ell\}) &= \frac{r(r+3)}{2\cdot N^2} \cdot \sum_{i=1}^{N}\sum_{j=1}^{N} P_2^{\frac{r-1}{2}}\left(\mathbf{x}_i \cdot \mathbf{x}_j\right) \\ &= \sum_{k=1}^{\frac{(r+3)r}{2}}\left[\frac{1}{N}\cdot\sum_{i=1}^{N} Y_{2k}^{(r)}(\mathbf{x}_i)\right]^2 \end{aligned}$$

that can be obtained by taking $A_2^2 = 1$ and $A_i^2 = +\infty$ for $i \neq 2$. Bingham's statistic converges to 0 for any sequence of points that is 1-uniformly distributed for functions $f \in H$, where

$$H = \left\{ f : \mathbb{S}^r \to \mathbb{R} \,\middle|\, \left|\widehat{f}_{01}^{(r)}\right|^2 + \sum_{k=1}^{\frac{(r+3)r}{2}} \left|\widehat{f}_{2k}^{(r)}\right|^2 < +\infty, \widehat{f}_{\ell k}^{(r)} = 0 \text{ for } \ell \neq 2 \right\}$$
$$= \mathcal{H}_0\left(\mathbb{S}^r\right) \oplus \mathcal{H}_2\left(\mathbb{S}^r\right).$$

3.3. *Worst-case error of quadrature rules*

For a certain $f \in H$, we define the integral

$$I(f) \triangleq \int_{\mathbb{S}^r} f(\mathbf{y})\, \mathrm{d}\omega_r^\star(\mathbf{y})$$

and a numerical integration rule

$$Q_N(f) = \sum_{j=1}^N w_j \cdot f\left(\mathbf{x}_j\right)$$

with $\sum_{j=1}^N w_j = 1$. The worst-case error for all $f \in \mathsf{B}(H)$ is

$$e(Q_N) \triangleq \sup\left\{|I(f) - Q_N(f)| : \|f\|_H \leq 1\right\}.$$

The following theorem gives an expression for $e(Q_N)$ analogous to the ones in [48, Section 4], limited to the case $r = 2$, and [7, Section 3.3], limited to the case $A_\ell^2 < +\infty$ for $\ell \in \mathbb{N}$.

Theorem 8. *Let H be the Hilbert space of Theorem 4. Let Q_N be a numerical integration rule with $w_j \geq 0$ for $j = 1, \ldots, N$ and $\sum_{j=1}^N w_j = 1$. Then*

$$e(Q_N) = \left[\sum_{j=1}^N \sum_{i=1}^N w_j w_i \cdot \sum_{\ell=1}^\infty \frac{N(r-1,\ell)}{A_\ell^2} \cdot P_\ell^{\frac{r-1}{2}}\left(\mathbf{x}_j \cdot \mathbf{x}_i\right)\right]^{1/2}.$$

For a quasi-Monte Carlo numerical integration rule (i.e. with $w_j = \frac{1}{N}$ for $j = 1, \ldots, N$)

$$e(Q_N) = D\left(\mathbf{P}_N; \{A_\ell\}\right).$$

The same reasoning can be used to extend Theorem 4.1 in [48], dealing with positive weight numerical integration rules. In contrast with the notation previously used, here $\mathcal{Q}_n$ indicates a positive weight numerical integration rule on $\mathbb{S}^r$ which integrates exactly all polynomials of degree at most n, i.e. any $p \in \mathcal{P}_n(\mathbb{S}^r)$. Then, it can be shown that

$$e(\mathcal{Q}_n) \leq \sqrt{\sum_{\ell=n+1}^\infty \frac{N(r-1,\ell)}{A_\ell^2}}.$$

If we suppose that $A_\ell^2 \sim c_r \cdot \ell^\alpha$ with $\alpha > r$ and using the fact that $N(r-1,\ell) \sim \frac{2\ell^{r-1}}{(r-1)!}$, for large n the bound behaves as

$$e(\mathcal{Q}_n) \lesssim \sqrt{\frac{2}{c_r (r-1)! (\alpha - r)}} \cdot n^{\frac{r-\alpha}{2}}.$$

This bound, like the one of Theorem 4.1 in [48], greatly overestimates the error. A proof of this fact can be obtained from the results in [9], where a non-improvable formula for the rate of decrease of the error (without leading constants) is provided (see also [10], for explicit examples of sequences achieving the optimal bound, and [6], for the case of more general compact Riemannian manifolds). Notice that also the bounds for extremal fundamental systems on $\mathbb{S}^2$ in Eqs. (2.8) and (2.9) in [48] can be extended to the hypersphere $\mathbb{S}^r$ using this framework (see [42, pp. 237–238] for the statement of the bounds in the general case).

3.4. Tractability results for multivariate integration

The results of the previous section can be used to derive some conditions for tractability of integration on the sphere. In the following we will consider integration rules with nonnegative weights, encompassing quasi-Monte Carlo numerical integration rules as a special case. Since tractability involves sequences of spaces of functions depending on the dimension of the space, we will explicitly indicate r in the following formulas, writing $H^{(r)}, K^{(r)}, \widehat{f}_{\ell k}^{(r)}, \mathbf{A}^{(r)}, A_\ell^{(r)}$ and $Q_{N,r}$ instead of $H, K, \widehat{f}_{\ell k}, \mathbf{A}, A_\ell$ and Q_N. For $N = 0$, we formally define $e\left(Q_{0,r}\right) = \sup\left\{|I(f)| : f \in H^{(r)}, \|f\|_{H^{(r)}} \leq 1\right\}$. Tractability is defined in terms of the worst-case error $e\left(Q_{N,r}\right)$ for all $f \in \mathsf{B}\left(H^{(r)}\right)$, through the quantity

$$N_{\min}^{(\varepsilon,r)} \triangleq \min\left\{N : \exists Q_{N,r} \text{ such that } e\left(Q_{N,r}\right) \leq \varepsilon \cdot e\left(Q_{0,r}\right)\right\}.$$

Definition 9. We say that multivariate integration in the sequence of spaces $H^{(r)}$ is tractable if there exist nonnegative C, α and β such that

$$N_{\min}^{(\varepsilon,r)} \leq C r^{\alpha} \varepsilon^{-\beta}$$

holds for all dimensions $r = 1, 2, \ldots$ and for all $\varepsilon \in (0, 1)$.

We say that multivariate integration in the sequence of spaces $H^{(r)}$ is strongly tractable if the inequality above holds with $\alpha = 0$.

The minimal (infimum) α and β are respectively called the r-exponent and the ε-exponent of (strong) tractability.

The following theorem can be established.

Theorem 10. *Consider the space*

$$H^{(r)} = \left\{ f : \mathbb{S}^r \to \mathbb{R} \left| \sum_{\ell=0}^{+\infty} \sum_{k=1}^{N(r-1,\ell)} \left(A_\ell^{(r)}\right)^2 \left|\widehat{f}_{\ell k}^{(r)}\right|^2 < +\infty, \widehat{f}_{\ell k}^{(r)} = 0 \text{ if } \left(A_\ell^{(r)}\right)^2 = +\infty \right. \right\},$$

where it is intended that $A_0^{(r)} = 1$. Suppose moreover that $\sum_{\ell=1}^{+\infty} \left(A_\ell^{(r)}\right)^{-2} \cdot N(r-1,\ell) < +\infty$ for every r. Then $e\left(Q_{0,r}\right) = 1$.

(i) *Integration in the sequence of spaces $H^{(r)}$ is strongly tractable with ε-exponent of strong tractability at most equal to 2 if*

$$\limsup_r \sum_{\ell=1}^{+\infty} \left(A_\ell^{(r)}\right)^{-2} \cdot N(r-1,\ell) < +\infty.$$

Let

$$\alpha \triangleq \limsup_r \frac{\ln\left[\sum_{\ell=1}^{+\infty} \left(A_\ell^{(r)}\right)^{-2} \cdot N(r-1,\ell)\right]}{\ln r}.$$

If $\alpha < +\infty$, integration in the sequence of spaces $H^{(r)}$ is tractable with ε-exponent and r-exponent of tractability respectively at most equal to 2 and α.

(ii) *Suppose that there exist $r_0 \in \mathbb{N}$ and $n, m \in \mathbb{N} \cup \{+\infty\}$ such that, for $r \geq r_0$, $\left(A_\ell^{(r)}\right)^2 < +\infty$ for any $\ell \in I$ and $\left(A_\ell^{(r)}\right)^2 = +\infty$ for any $\ell \notin I$, where $I \triangleq \{\ell \in \mathbb{N}_0 | 0 \leq \ell \leq n, \ell = i \cdot m \text{ for } i \in \mathbb{N}_0\}$. If*

$$\liminf_{r} \inf_{\ell\in I} \min_{j\in I, 0\le j\le \ell} \frac{N(r-1,j)\left(A_\ell^{(r)}\right)^2}{N(r-1,\ell)\left(A_j^{(r)}\right)^2} > 0 \tag{6}$$

then the conditions stated in (i) *are also necessary.*

Remark 11. (i) Suppose that the sequence $\left\{A_\ell^{(r)}\right\}$ is the symbol of a pseudodifferential operator $\mathbf{A}^{(r)}$. The condition $\sum_{\ell=1}^{+\infty}\left(A_\ell^{(r)}\right)^{-2}\cdot N(r-1,\ell) < +\infty$ requires $\mathbf{A}^{(r)}$ to be at least of order t with $t > r/2$. Therefore, the order t of the operator $\mathbf{A}^{(r)}$ must increase with r.

(ii) Relation (6) is automatically satisfied if

$$\frac{N(r-1,i\cdot m)/\left(A_{i\cdot m}^{(r)}\right)^2}{N(r-1,(i+1)\cdot m)/\left(A_{(i+1)\cdot m}^{(r)}\right)^2} \ge 1$$

for any $i\in\mathbb{N}_0$ and any $r\ge r_0$.

(iii) Consider the space $H^{(r)}$ obtained by taking $A_0^{(r)}=1$ and $A_\ell^{(r)}=+\infty$ for $\ell>1$ and any r. Condition (6) becomes $\liminf_r \frac{\left(A_1^{(r)}\right)^2}{r+1} > 0$. Under this condition, strong tractability with ε-exponent of strong tractability at most equal to 2 always holds. On the other hand, integration in the space $H^{(r)}$ is tractable iff $\liminf_r \frac{\ln\left(A_1^{(r)}\right)^2}{\ln r} > -\infty$ with d-exponent at most $\alpha = 1-\liminf_r \frac{\ln\left(A_1^{(r)}\right)^2}{\ln r}$. Now, consider the space $H^{(r)}$ obtained by taking $A_0^{(r)}=1$ and $A_\ell^{(r)}=+\infty$ for any other $\ell\neq 2$. Condition (6) becomes $\liminf_r \frac{\left(A_2^{(r)}\right)^2}{(r+3)r} > 0$. Then, strong tractability with ε-exponent of strong tractability at most equal to 2 always holds. Iff

$$\alpha = 2-\liminf_r \frac{\ln\left(A_2^{(r)}\right)^2}{\ln r} < +\infty,$$

tractability with d-exponent at most α holds.

3.5. Koksma–Hlawka inequalities

In this section, we present several inequalities in the Koksma–Hlawka style concerning the error of equal-weight numerical integration. The following Koksma–Hlawka inequality for functions belonging to a Hilbert space H allows us to obtain as a corollary an inequality for functions belonging to a Sobolev class that generalizes the one in [12].

Proposition 12. *Consider a function f belonging to the Hilbert space H, endowed with the inner product $(\cdot,\cdot)_H$ and the norm $\|\cdot\|_H$, as defined in Theorem 4. Then, the following inequality holds*

$$\left|\int_{\mathbb{S}^r} f(\mathbf{x})\,\mathrm{d}\omega_r^\star(\mathbf{x}) - \frac{1}{N}\sum_{i=1}^N f(\mathbf{x}_i)\right| \le \|f\|_H\cdot D\left(\mathbf{P}_N;\{A_\ell\}\right).$$

Corollary 13. *Let $\mathbf{A}$ be a pseudodifferential operator of order s, where $s>\frac{r}{2}$, with symbol $\{A_\ell\}_{\ell\in\mathbb{N}_0}$ such that $A_\ell\neq 0$, for $\ell\ge 1$. Then, for any function $f\in H^s(\mathbb{S}^r)$, we have the following estimate:*

$$\left|\int_{\mathbb{S}^r} f(\mathbf{x})\,\mathrm{d}\omega_r^\star(\mathbf{x}) - \frac{1}{N}\sum_{i=1}^N f(\mathbf{x}_i)\right| \le \|\mathbf{A}f\|_{L_2}\cdot D\left(\mathbf{P}_N;\{A_\ell\}\right).$$

Remark 14. Using (2) and applying Corollary 13 to the centered version of f (namely $f - \int_{\mathbb{S}^r} f(\mathbf{x})\, \mathrm{d}\omega_r^\star(\mathbf{x})$), in the previous formula $\|\mathbf{A}f\|_{L_2}$ can be replaced by $\left\|\mathbf{A}\left(f - \int_{\mathbb{S}^r} f(\mathbf{x})\, \mathrm{d}\omega_r^\star(\mathbf{x})\right)\right\|_{L_2}$.

The following Koksma–Hlawka inequalities require that the sequence of weights $\{A_\ell\}_{\ell\in\mathbb{N}}$ is modified so that $A_\ell = +\infty$ for any $\ell > n$; to stress this fact, we write $D_{(n)}(\mathbf{P}_N; \{A_\ell\})$ instead of $D(\mathbf{P}_N; \{A_\ell\})$, where

$$D_{(n)}(\mathbf{P}_N; \{A_\ell\}) = \left[\sum_{\ell=1}^{n} \frac{1}{A_\ell^2} \cdot \sum_{k=1}^{N(r-1,\ell)} \left|\frac{1}{N} \cdot \sum_{i=1}^{N} Y_{\ell k}^{(r)}(\mathbf{x}_i)\right|^2\right]^{\frac{1}{2}}.$$

By the way, using this definition, it is easy to see that $D_{(n)}(\mathbf{P}_N; \{A_\ell\}) \leq D_{(n+1)}(\mathbf{P}_N; \{A_\ell\})$, so that in the following formulas $D_{(n)}(\mathbf{P}_N; \{A_\ell\})$ could be majorized by $D(\mathbf{P}_N; \{A_\ell\})$.

Concerning our next results, Proposition 15 is restricted to polynomials, Proposition 17 extends the previous bound to general functions respecting a Lipschitz-type condition on the sphere while Proposition 19 concerns indicator functions of K-regular sets (see below for a definition).

Proposition 15. *Let f be a polynomial of degree at most $n \geq 1$ on $\mathbb{S}^r$, and $\{A_\ell\}_{\ell\in\mathbb{N}}$ a sequence of weights. Then, the following inequality holds:*

$$\left|\int_{\mathbb{S}^r} f(\mathbf{x})\, \mathrm{d}\omega_r^\star(\mathbf{x}) - \frac{1}{N}\sum_{i=1}^{N} f(\mathbf{x}_i)\right| \leq \max_{1\leq\ell\leq n} |A_\ell| \cdot \left\|f - \int_{\mathbb{S}^r} f(\mathbf{x})\, \mathrm{d}\omega_r^\star(\mathbf{x})\right\|_{L_2} \cdot D_{(n)}(\mathbf{P}_N; \{A_\ell\}).$$

Remark 16. In the previous formula (as in the following ones), it is possible to use the following trivial majorization:

$$\left\|f - \int_{\mathbb{S}^r} f(\mathbf{x})\, \mathrm{d}\omega_r^\star(\mathbf{x})\right\|_{L_2} \leq \|f\|_{L_2}.$$

Proposition 17. *Let f be a continuous function on $\mathbb{S}^r$ such that*

$$|f(\mathbf{x}) - f(\mathbf{y})| \leq C_f \cdot \arccos(\mathbf{x}\cdot\mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \mathbb{S}^r,$$

for a constant $C_f > 0$. Then for any positive integer m

$$\left|\int_{\mathbb{S}^r} f(\mathbf{x})\, \mathrm{d}\omega_r^\star(\mathbf{x}) - \frac{1}{N}\sum_{i=1}^{N} f(\mathbf{x}_i)\right| \leq \frac{\sqrt{20}rC_f}{m} + \max_{1\leq\ell\leq 2m} |A_\ell| \cdot \|f\|_{\mathcal{C}} \cdot D_{(2m)}(\mathbf{P}_N; \{A_\ell\}).$$

Remark 18. (i) This result should be compared with Theorem 1 in [22], that constitutes an Erdős–Turán type inequality. By the way, using Remark 3 on p. 332 in [22], if $w(\cdot)$ is a function satisfying $|f(\mathbf{x}) - f(\mathbf{y})| \leq w(\arccos(\mathbf{x}\cdot\mathbf{y}))$ for $\mathbf{x}, \mathbf{y} \in \mathbb{S}^r$, we have for any positive integer m

$$\left|\int_{\mathbb{S}^r} f(\mathbf{x})\, \mathrm{d}\omega_r^\star(\mathbf{x}) - \frac{1}{N}\sum_{i=1}^{N} f(\mathbf{x}_i)\right| \leq 7 \cdot w\left(\frac{r}{m}\right) + \max_{1\leq\ell\leq 2m} |A_\ell| \cdot V(f) \cdot D_{(2m)}(\mathbf{P}_N; \{A_\ell\}),$$

where $V(f) = \sup_{\mathbf{x},\mathbf{y}\in\mathbb{S}^r} |f(\mathbf{x}) - f(\mathbf{y})|$.

(ii) Theorem 1 and 2 in [13] can be obtained as corollaries of our results under the conditions used in the original paper, taking

$$A_\ell = \frac{N(r-1,\ell)}{\omega_r^{1/2} \cdot b_\ell},$$

where b_ℓ is defined on p. 240 of [13].

We will need the following class of sets, introduced in [47]. A measurable set $B \subset \mathbb{S}^r$ is said to be K-*regular* if $\omega_r^\star(\partial_\delta B) \leq K\delta$ for $\delta > 0$, where $\partial_\delta B \triangleq \{y \in \mathbb{S}^r : d(y, B) \leq \delta, d(y, \mathbb{S}^r \setminus B) \leq \delta\}$ and d is the Euclidean distance. As an example, spherical caps are K_0-regular with a constant K_0 depending only on the dimension r of the space, while rectifiable curves on the sphere having length ℓ are $(K_0 + \ell \cdot K_1)$-regular for constants K_0 and K_1 depending only on the dimension r of the space.

Proposition 19. *Let f be the indicator function of a K-regular set. There exists $C_0 > 0$ depending only on r such that for any integer n*

$$\left|\int_{\mathbb{S}^r} f(\mathbf{x})\, d\omega_r^\star(\mathbf{x}) - \frac{1}{N}\sum_{i=1}^{N} f(\mathbf{x}_i)\right| \leq C_0 \cdot \left\{\frac{K \cdot \left\lceil \frac{r+3}{2} \right\rceil}{n} + \max_{1\leq \ell \leq n} |A_\ell| \cdot D_{(n)}(\mathbf{P}_N; \{A_\ell\})\right\},$$

where $\lceil x \rceil$ is the smallest integer not less than x.

Remark 20. This provides a rigorous version of Eq. (13) in [12]. This should be compared to the Erdős–Turán inequalities of [27], [21, Theorem 1], [32, Theorem 5], where explicit constants are provided for the special case of the indicator function of a spherical cap. Our result is inspired by the inequality between discrepancy and polynomial quality of integration of [2, Theorem 1].

4. Proofs

Proof of Theorem 4. We start by considering a sequence of weights $\{B_\ell\}_{\ell \in \mathbb{N}_0}$ with $B_0 = 1$ and $B_\ell > 0$ for any $\ell \in \mathbb{N}$; we also allow $B_\ell \in \overline{\mathbb{R}}_+$. We adopt the convention that $0 \cdot \infty = 0$ (the derivation can be made more rigorous excluding from the sums the indexes for which $B_\ell = +\infty$). Suppose moreover that $\sum_{\ell=0}^{\infty} \frac{N(r-1,\ell)}{B_\ell} < +\infty$. We define the Hilbert space

$$H \triangleq \left\{ f : \mathbb{S}^r \to \mathbb{R} \left| \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(r-1,\ell)} B_\ell \left| \widehat{f}_{\ell k}^{(r)} \right|^2 < +\infty \right.\right\}$$

with inner product

$$(f, g)_H \triangleq \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(r-1,\ell)} B_\ell \widehat{f}_{\ell k}^{(r)} \widehat{g}_{\ell k}^{(r)},$$

where $\widehat{f}_{\ell k}^{(r)}$ is the Fourier coefficient of f corresponding to $Y_{\ell k}^{(r)}$. We also consider the corresponding norm $\|f\|_H \triangleq (f, f)_H^{1/2}$. The kernel K associated with the Hilbert space H and the inner product $(f, g)_H$ is

$$K(\mathbf{x}, \mathbf{y}) \triangleq \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(r-1,\ell)} \frac{1}{B_\ell} \cdot Y_{\ell k}^{(r)}(\mathbf{x}) Y_{\ell k}^{(r)}(\mathbf{y}) = \sum_{\ell=0}^{\infty} \frac{N(r-1,\ell)}{B_\ell} \cdot P_\ell^{\frac{r-1}{2}}(\mathbf{x} \cdot \mathbf{y}). \tag{7}$$

The assumption $\sum_{\ell=0}^{\infty} \frac{N(r-1,\ell)}{B_\ell} < +\infty$ we made before ensures the existence of this sum for all $\mathbf{x}, \mathbf{y} \in \mathbb{S}^r$. This is the reproducing kernel in H since for each $f \in H$ and $\mathbf{y} \in \mathbb{S}^r$

$$K(\cdot, \mathbf{y}) \in H, \quad (f, K(\cdot, \mathbf{y}))_H = f(\mathbf{y}).$$

We define the quantity

$$r_N(\mathbf{y}) \triangleq \frac{1}{N} \sum_{j=1}^{N} K\left(\mathbf{y}, \mathbf{x}_j\right) - 1.$$

The diaphony in the sense of [1, p. 501] is given by

$$\|r_N\|_H^2 = (r_N, r_N)_H = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(r-1,\ell)} B_\ell \cdot \left\{ \widehat{[r_N]}_{\ell k}^{(r)} \right\}^2 ,$$

where $\left\{ \widehat{[r_N]}_{\ell k}^{(r)} \right\}_{\ell,k}$ are the Fourier coefficients defined as in (1). In order to identify the Fourier coefficients, we use (7):

$$\begin{aligned} r_N(\mathbf{y}) &= \frac{1}{N} \sum_{j=1}^{N} K(\mathbf{y}, \mathbf{x}_j) - 1 = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(r-1,\ell)} \frac{\left[\frac{1}{N} \sum_{j=1}^{N} Y_{\ell k}^{(r)}(\mathbf{x}_j) \right]}{B_\ell} \cdot Y_{\ell k}^{(r)}(\mathbf{y}) - 1 \\ &= \sum_{\ell=1}^{\infty} \sum_{k=1}^{N(r-1,\ell)} \frac{\left[\frac{1}{N} \sum_{j=1}^{N} Y_{\ell k}^{(r)}(\mathbf{x}_j) \right]}{B_\ell} \cdot Y_{\ell k}^{(r)}(\mathbf{y}) \end{aligned}$$

and we finally get

$$\widehat{[r_N]}_{\ell k}^{(r)} = \int_{\mathbb{S}^r} r_N(\mathbf{x}) Y_{\ell k}^{(r)}(\mathbf{x}) \, \mathrm{d}\omega_r^\star(\mathbf{x}) = \begin{cases} 0 & \ell = 0 \\ \dfrac{\left[\frac{1}{N} \sum_{j=1}^{N} Y_{\ell k}^{(r)}(\mathbf{x}_j) \right]}{B_\ell} & \ell \geq 1, \end{cases}$$

where it is intended that $\widehat{[r_N]}_{\ell k}^{(r)} = 0$ whenever $B_\ell = +\infty$. Therefore

$$\begin{aligned} \|r_N\|_H^2 &= (r_N, r_N)_H = B_0 \cdot \left\{ \widehat{[r_N]}_{01}^{(r)} \right\}^2 + \sum_{\ell=1}^{\infty} \sum_{k=1}^{N(r-1,\ell)} B_\ell \cdot \left\{ \widehat{[r_N]}_{\ell k}^{(r)} \right\}^2 \\ &= \sum_{\ell=1}^{\infty} \sum_{k=1}^{N(r-1,\ell)} \frac{\left[\frac{1}{N} \sum_{j=1}^{N} Y_{\ell k}^{(r)}(\mathbf{x}_j) \right]^2}{B_\ell} . \end{aligned}$$

This is exactly $D^2(\mathbf{P}_N; \{A_\ell\})$ when the sequence of weights $\{A_\ell\}_{\ell \in \mathbb{N}}$ respects the equality $B_\ell = A_\ell^2$ for every $\ell \geq 1$. □

Proof of Proposition 6. (i) The result is a simple application of Theorem 3 of [1] to the reproducing kernel Hilbert space identified in Theorem 4.

(ii) If we consider the uniform probability measure $\omega_r^\star(\cdot)$ on the sphere $\mathbb{S}^r$, the function g respecting Proposition 6 in [1] is $g \equiv 1$. Therefore, Theorem 7(2) applies with $g \equiv 1$: in practice 1-uniform distribution is equivalent to uniform distribution with respect to the uniform probability measure $\omega_r^\star(\cdot)$ on $\mathbb{S}^r$ (see p. 506 of [1] for a definition), if and only if $\overline{\mathrm{lin}\{H, 1\}} = \mathcal{C}(\mathbb{S}^r)$. Now, this is true if and only if all of the coefficients A_ℓ are finite. □

Proof of Theorem 8. The reproducing kernel property of K allows us to write the integration error for f as

$$I(f) - Q_N(f) = (f, \xi)_H ,$$

where ξ, called the *representer* of the error, is

$$\xi(\mathbf{x}) = \int_{\mathbb{S}^r} K(\mathbf{x}, \mathbf{y}) \, \mathrm{d}\omega_r^\star(\mathbf{y}) - \sum_{j=1}^{N} w_j \cdot K(\mathbf{x}, \mathbf{x}_j) .$$

The integration error can be written as

$$\begin{aligned} e\left(Q_N\right) &= \sup\left\{\left|(f,\xi)_H\right| \mid \|f\|_H \le 1\right\} = \|\xi\|_H \\ &= \left[\int_{\mathbb{S}^r}\left(\int_{\mathbb{S}^r} K\left(\mathbf{x},\mathbf{y}\right) \mathrm{d}\omega_r^\star\left(\mathbf{x}\right)\right) \mathrm{d}\omega_r^\star\left(\mathbf{y}\right) - 2\sum_{j=1}^N w_j \cdot \int_{\mathbb{S}^r} K\left(\mathbf{x},\mathbf{x}_j\right) \mathrm{d}\omega_r^\star\left(\mathbf{x}\right) \right. \\ &\quad \left. + \sum_{j=1}^N \sum_{i=1}^N w_j w_i \cdot K\left(\mathbf{x}_j,\mathbf{x}_i\right)\right]^{1/2}. \end{aligned}$$

Using the fact that $\int_{\mathbb{S}^r} K\left(\mathbf{x},\mathbf{y}\right) \mathrm{d}\omega_r^\star\left(\mathbf{x}\right) = 1$ (provided that $A_0 = 1$ as in Theorem 4), for any $\mathbf{y} \in \mathbb{S}^r$, we get

$$e\left(Q_N\right) = \left[-1 + \sum_{j=1}^N \sum_{i=1}^N w_j w_i \cdot K\left(\mathbf{x}_j,\mathbf{x}_i\right)\right]^{1/2},$$

where we have used the fact that $\sum_{j=1}^N w_j = 1$. Now, we have

$$\begin{aligned} e\left(Q_N\right) &= \left[-1 + \sum_{j=1}^N \sum_{i=1}^N w_j w_i \cdot K\left(\mathbf{x}_j,\mathbf{x}_i\right)\right]^{1/2} \\ &= \left[-1 + \sum_{j=1}^N \sum_{i=1}^N w_j w_i \cdot \sum_{\ell=0}^\infty \frac{N\left(r-1,\ell\right)}{A_\ell^2} \cdot P_\ell^{\frac{r-1}{2}}\left(\mathbf{x}_j \cdot \mathbf{x}_i\right)\right]^{1/2} \\ &= \left[-1 + \sum_{j=1}^N \sum_{i=1}^N w_j w_i \cdot \frac{N\left(r-1,0\right)}{A_0^2} \cdot P_0^{\frac{r-1}{2}}\left(\mathbf{x}_j \cdot \mathbf{x}_i\right) \right. \\ &\quad \left. + \sum_{j=1}^N \sum_{i=1}^N w_j w_i \cdot \sum_{\ell=1}^\infty \frac{N\left(r-1,\ell\right)}{A_\ell^2} \cdot P_\ell^{\frac{r-1}{2}}\left(\mathbf{x}_j \cdot \mathbf{x}_i\right)\right]^{1/2} \\ &= \left[\sum_{j=1}^N \sum_{i=1}^N w_j w_i \cdot \sum_{\ell=1}^\infty \frac{N\left(r-1,\ell\right)}{A_\ell^2} \cdot P_\ell^{\frac{r-1}{2}}\left(\mathbf{x}_j \cdot \mathbf{x}_i\right)\right]^{1/2}, \end{aligned} \tag{8}$$

where we use again the fact that $\sum_{j=1}^N w_j = 1$. Then, if the weights are taken to be equal (namely, $w_j = \frac{1}{N}$ for $1 \le j \le N$) we eventually get

$$e\left(Q_N\right) = D\left(\mathbf{P}_N; \{A_\ell\}\right). \quad \square$$

Proof of Theorem 10. Since the class of QMC quadrature rules is included in the class of integration rules with nonnegative weights, $N_{\min}^{(\varepsilon,r)}$ is smaller for the latter than for the former and an upper bound can be established considering only the case of QMC rules.

The worst-case error of integration for all $f \in \mathrm{B}\left(H^{(r)}\right)$ is given by $e\left(Q_{N,r}\right) = D\left(\mathbf{P}_N; \left\{A_\ell^{(r)}\right\}\right)$. When $N = 0$, the error of integration is

$$e\left(Q_{0,r}\right) = \left[\int_{\mathbb{S}^r}\left(\int_{\mathbb{S}^r} K^{(r)}\left(\mathbf{x},\mathbf{y}\right) \mathrm{d}\omega_r^\star\left(\mathbf{x}\right)\right) \mathrm{d}\omega_r^\star\left(\mathbf{y}\right)\right]^{1/2} = 1,$$

where we have used the fact that $\int_{\mathbb{S}^r} K^{(r)}\left(\mathbf{x},\mathbf{y}\right) \mathrm{d}\omega_r^\star\left(\mathbf{x}\right) = 1$ since $A_0^{(r)} = 1$.

(i) In order to prove tractability, we reason as in [49, pp. 15–16] and we prove instead tractability for average sample points. The authors show that this is enough since the d- and ε-exponents for tractability of QMC quadrature rules are not greater than the corresponding ones for tractability for average sample points. Therefore, for all N and all r, we define $\mathrm{av}_2(N,r) \triangleq \left\{\mathbb{E}\left[e\left(Q_{N,r}\right)\right]^2\right\}^{1/2}$, where $\mathbb{E}$ denotes the expectation with respect to the sample $\mathbf{P}_N$. Then, we obtain

$$\begin{aligned}\mathrm{av}_2(N,r) &= \left\{\mathbb{E}\left[\frac{1}{N^2}\cdot\sum_{j=1}^{N}\sum_{i=1}^{N}\sum_{\ell=1}^{\infty}\frac{N(r-1,\ell)}{\left(A_\ell^{(r)}\right)^2}\cdot P_\ell^{\frac{r-1}{2}}\left(\mathbf{x}_j\cdot\mathbf{x}_i\right)\right]\right\}^{1/2}\\ &= \left[\frac{1}{N}\cdot\sum_{\ell=1}^{+\infty}\frac{N(r-1,\ell)}{\left(A_\ell^{(r)}\right)^2}\right]^{1/2},\end{aligned}$$

where we have used the fact that $\mathbb{E}\left[P_\ell^{\frac{r-1}{2}}\left(\mathbf{x}_j\cdot\mathbf{x}_i\right)\right]=\delta_{ij}$ for $\ell\geq 1$. We get

$$N_{\min}^{(\varepsilon,r)} \leq \varepsilon^{-2}\cdot\sum_{\ell=1}^{+\infty}\left(A_\ell^{(r)}\right)^{-2}\cdot N(r-1,\ell) = \varepsilon^{-2}\cdot r^{\frac{\ln\sum_{\ell=1}^{+\infty}\left(A_\ell^{(r)}\right)^{-2}\cdot N(r-1,\ell)}{\ln r}}.$$

Tractability with exponent α holds when

$$\alpha \triangleq \limsup_r \frac{\ln\left[\sum_{\ell=1}^{+\infty}\left(A_\ell^{(r)}\right)^{-2}\cdot N(r-1,\ell)\right]}{\ln r} < +\infty,$$

strong tractability when

$$\limsup_r \sum_{\ell=1}^{+\infty}\left(A_\ell^{(r)}\right)^{-2}\cdot N(r-1,\ell) < +\infty.$$

(ii) First of all, we minorize $e\left(Q_{N,r}\right)^2$ as

$$\begin{aligned}e\left(Q_{N,r}\right)^2 &= -1+\frac{1}{N^2}\cdot\sum_{j=1}^{N}\sum_{i=1}^{N}\sum_{\ell\in I}\frac{N(r-1,\ell)}{\left(A_\ell^{(r)}\right)^2}\cdot P_\ell^{\frac{r-1}{2}}\left(\mathbf{x}_j\cdot\mathbf{x}_i\right)\\ &= -1+\sum_{\ell\in I}\frac{1}{\left(A_\ell^{(r)}\right)^2}\cdot\sum_{k=1}^{N(r-1,\ell)}\left|\frac{1}{N}\cdot\sum_{i=1}^{N}Y_{\ell k}^{(r)}\left(\mathbf{x}_i\right)\right|^2\\ &\geq -1+\sum_{\ell\in I}\frac{1}{N(r-1,\ell)}\cdot a_\ell^{(r)}\cdot\sum_{k=1}^{N(r-1,\ell)}\left|\frac{1}{N}\cdot\sum_{i=1}^{N}Y_{\ell k}^{(r)}\left(\mathbf{x}_i\right)\right|^2\\ &\geq -1+\frac{1}{N^2}\cdot\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\ell\in I}a_\ell^{(r)}\cdot P_\ell^{\frac{r-1}{2}}\left(\mathbf{x}_i\cdot\mathbf{x}_j\right),\end{aligned}$$

where $a_\ell^{(r)} \triangleq \min_{j\in I,0\leq j\leq\ell} N(r-1,j)/\left(A_j^{(r)}\right)^2$. Then we separate in this last term the contributions of the diagonal elements from the extra-diagonal ones:

$$\begin{aligned}e\left(Q_{N,r}\right)^2 &\geq -1+\frac{1}{N^2}\cdot\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\ell\in I}a_\ell^{(r)}\cdot P_\ell^{\frac{r-1}{2}}\left(\mathbf{x}_i\cdot\mathbf{x}_j\right)\\ &= -1+\frac{1}{N}\cdot\sum_{\ell\in I}a_\ell^{(r)}+\frac{1}{N^2}\cdot\sum_{1\leq i\neq j\leq N}\sum_{\ell\in I}a_\ell^{(r)}\cdot P_\ell^{\frac{r-1}{2}}\left(\mathbf{x}_i\cdot\mathbf{x}_j\right).\end{aligned}$$

We claim that, under the condition that $\left\{a_\ell^{(r)}\right\}_{\ell\in I}$ is a (non strictly) decreasing sequence, $\sum_{\ell\in I} a_\ell^{(r)} \cdot P_\ell^{\frac{r-1}{2}}(x) > 0$ for $-1 < x < 1$ and $\sum_{\ell\in I} a_\ell^{(r)} \cdot P_\ell^{\frac{r-1}{2}}(x) \geq 0$, by continuity, for $-1 \leq x \leq 1$.

Indeed, let $\{b_\ell\}_{\ell\in\mathbb{N}_0}$ be a (non strictly) decreasing sequence of constants, $1 = b_0 \geq b_1 \geq \cdots$. Then, under the condition $\sum_{\ell=0}^{\infty} |b_\ell| < +\infty$, $\sum_{\ell=0}^{\infty} b_\ell P_{\ell m}^{\lambda}(x) > 0$ for $-1 < x < 1$. Indeed, provided $\sum_{\ell=0}^{\infty} b_\ell P_{\ell m}^{\lambda}(x)$ is convergent for $-1 < x < 1$, we can write

$$\sum_{\ell=0}^{\infty} b_\ell P_{\ell m}^{\lambda}(x) = \sum_{\ell=0}^{\infty} (b_\ell - b_{\ell+1}) \cdot \sum_{k=0}^{\ell} P_{km}^{\lambda}(x), \quad -1 < x < 1.$$

For $\lambda > m/2$, Remark 5 in [45] ensures that $\sum_{k=0}^{\ell} P_{km}^{\lambda}(x) > 0$ for $-1 < x < 1$ and any $\ell \in \mathbb{N}_0$.[4] The condition $\sum_{\ell=0}^{\infty} |b_\ell| < +\infty$ implies that convergence (indeed, uniform convergence) of $\sum_{\ell=0}^{\infty} b_\ell P_{\ell m}^{\lambda}(x)$ over $-1 < x < 1$ holds. Then, the fact that $b_\ell \geq b_{\ell+1}$ for any $\ell \in \mathbb{N}_0$ implies that also $\sum_{\ell=0}^{\infty} b_\ell P_{\ell m}^{\lambda}(x) > 0$ for $-1 < x < 1$.

Now, we identify λ and $\{b_\ell\}_{\ell\in\mathbb{N}_0}$ respectively with $\frac{r-1}{2}$ and $\left\{a_0^{(r)}, a_m^{(r)}, \ldots\right\}$.[5] The condition $\sum_{\ell=0}^{\infty} |b_\ell| < +\infty$ becomes $\sum_{\ell\in I} a_\ell^{(r)} < +\infty$ and this is implied by the requirement $\sum_{\ell=1}^{+\infty} \left(A_\ell^{(r)}\right)^{-2} \cdot N(r-1,\ell) < +\infty$ in the statement of the theorem. For $r \geq m+1$, we have $\sum_{\ell\in I} a_\ell^{(r)} \cdot P_\ell^{\frac{r-1}{2}}\left(\mathbf{x}_j \cdot \mathbf{x}_i\right) \geq 0$ for every $1 \leq i,j \leq N$ with $i \neq j$ and

$$e\left(Q_{N,r}\right)^2 \geq -1 + \frac{1}{N} \cdot \sum_{\ell\in I} a_\ell^{(r)}.$$

From $e\left(Q_{N_{\min}^{(\varepsilon,r)},r}\right) \leq \varepsilon \cdot e\left(Q_{0,r}\right)$, we get

$$\varepsilon^2 = \varepsilon^2 \cdot e\left(Q_{0,r}\right)^2 \geq e\left(Q_{N_{\min}^{(\varepsilon,r)},r}\right)^2 \geq -1 + \frac{1}{N_{\min}^{(\varepsilon,r)}} \cdot \sum_{\ell\in I} a_\ell^{(r)}$$

and

$$N_{\min}^{(\varepsilon,r)} \geq \frac{1}{1+\varepsilon^2} \cdot \sum_{\ell\in I} a_\ell^{(r)}.$$

Now, suppose that $r \geq r_0$. Then we can write

$$\begin{aligned}\sum_{\ell\in I} a_\ell^{(r)} &= \sum_{\ell\in I} \frac{N(r-1,\ell)}{\left(A_\ell^{(r)}\right)^2} \cdot \left\{\min_{j\in I, 0\leq j\leq \ell} \frac{N(r-1,j)\left(A_\ell^{(r)}\right)^2}{N(r-1,\ell)\left(A_j^{(r)}\right)^2}\right\} \\ &\geq \left\{\inf_{\ell\in I}\min_{j\in I, 0\leq j\leq \ell} \frac{N(r-1,j)\left(A_\ell^{(r)}\right)^2}{N(r-1,\ell)\left(A_j^{(r)}\right)^2}\right\} \cdot \sum_{\ell\in I} \frac{N(r-1,\ell)}{\left(A_\ell^{(r)}\right)^2}.\end{aligned}$$

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[4] The case with $m = 1$ has received a lot of attention in the literature. It was first proved in the case $\lambda = 0.5$ in [16], and then extended to $\lambda \geq 0.5$ in [17, p. 275] and to $\lambda > \lambda' \simeq 0.23061297$ in [29, inequality (6.9)].

[5] Here it is intended that, if the n appearing in the definition of I is finite, an infinite sequence of zeros will be added to the sequence of weights $a_\ell^{(r)}$.

Now we take logarithms of both sides and we use the fact that $\limsup_i (x_i + y_i) \geq \limsup_i x_i + \liminf_i y_i$, provided the sum is not of the form $+\infty - \infty$ (see, e.g., [44, p. 38]):

$$\begin{aligned}\limsup_r \ln N_{\min}^{(\varepsilon,r)} &\geq -\ln\left(1+\varepsilon^2\right) + \left\{\limsup_r \ln \sum_{\ell\in I} \frac{N(r-1,\ell)}{\left(A_\ell^{(r)}\right)^2}\right\} \\ &\quad + \left\{\liminf_r \ln \inf_{\ell\in I} \min_{j\in I, 0\leq j\leq \ell} \frac{N(r-1,j)\left(A_\ell^{(r)}\right)^2}{N(r-1,\ell)\left(A_j^{(r)}\right)^2}\right\} \\ &= -\ln\left(1+\varepsilon^2\right) + \ln\left\{\limsup_r \sum_{\ell\in I} \frac{N(r-1,\ell)}{\left(A_\ell^{(r)}\right)^2}\right\} \\ &\quad + \ln\left\{\liminf_r \inf_{\ell\in I} \min_{j\in I, 0\leq j\leq \ell} \frac{N(r-1,j)\left(A_\ell^{(r)}\right)^2}{N(r-1,\ell)\left(A_j^{(r)}\right)^2}\right\}.\end{aligned}$$

Under the condition that

$$\liminf_r \inf_{\ell\in I} \min_{j\in I, 0\leq j\leq \ell} \frac{N(r-1,j)\left(A_\ell^{(r)}\right)^2}{N(r-1,\ell)\left(A_j^{(r)}\right)^2} > 0,$$

the sum is not of the form $+\infty - \infty$. If

$$\limsup_r \sum_{\ell=1}^{+\infty} \left(A_\ell^{(r)}\right)^{-2} \cdot N(r-1,\ell) = +\infty,$$

$\limsup_r N_{\min}^{(\varepsilon,r)} = +\infty$ and strong tractability cannot hold. A similar reasoning can be repeated for tractability. □

Proof of Proposition 12. The result follows from the second Remark on p. 501 in [1] (see also [15, Theorem 5]) and the proof of Theorem 4. □

Proof of Corollary 13. A proof can be obtained following strictly the one of Theorem 3.1 in [12], but we obtain our result as a Corollary of Proposition 12. According to Remark 5, we know that $H^s(\mathbb{S}^r) = H$ whenever in the definition of H we take $c_1\ell^s \leq A_\ell \leq c_2\ell^s$ for $\ell \geq 1$ and $s > \frac{r}{2}$. This allows us to apply Proposition 12 for a function $f \in H^s(\mathbb{S}^r)$ with $s > \frac{r}{2}$; the norm $\|f\|_H$ appearing in the statement becomes then the norm $\|\mathbf{A}f\|_{L_2}$. □

Proof of Proposition 15. Since f is a polynomial of degree at most n, we can write f as

$$f(\mathbf{x}) = \int_{\mathbb{S}^r} f(\mathbf{x})\,\mathrm{d}\omega_r^\star(\mathbf{x}) + \sum_{\ell=1}^{n} \sum_{k=1}^{N(r-1,\ell)} \widehat{f}_{\ell k}^{(r)} \cdot Y_{\ell k}^{(r)}(\mathbf{x}).$$

We apply Proposition 12 to $f - \int_{\mathbb{S}^r} f(\mathbf{y})\,\mathrm{d}\omega_r^\star(\mathbf{y})$, getting

$$\left|\int_{\mathbb{S}^r} f(\mathbf{x})\,\mathrm{d}\omega_r^\star(\mathbf{x}) - \frac{1}{N}\sum_{i=1}^{N} f(\mathbf{x}_i)\right| \leq \left\| f - \int_{\mathbb{S}^r} f(\mathbf{x})\,\mathrm{d}\omega_r^\star(\mathbf{x})\right\|_H \cdot D_{(n)}(\mathbf{P}_N; \{A_\ell\}).$$

Using the expression for $\|f\|_H$ in Theorem 4, the following trivial majorization holds:

$$\left\| f - \int_{\mathbb{S}^r} f(\mathbf{x})\,\mathrm{d}\omega_r^\star(\mathbf{x})\right\|_H = \sum_{\ell=1}^{n} \sum_{k=1}^{N(r-1,\ell)} \left|\widehat{f}_{\ell k}^{(r)}\right|^2 \cdot |A_\ell|^2$$

$$\leq \max_{1\leq\ell\leq n} |A_\ell|^2 \cdot \sum_{\ell=1}^{n} \sum_{k=1}^{N(r-1,\ell)} \left|\widehat{f}_{\ell k}^{(r)}\right|^2$$
$$= \max_{1\leq\ell\leq n} |A_\ell|^2 \cdot \left\| f - \int_{\mathbb{S}^r} f(\mathbf{x})\, \mathrm{d}\omega_r^\star(\mathbf{x}) \right\|_{L_2}^2 .$$

The proposition can also be proved using a variant of Theorem 1 in [13] (see also [14]). □

Proof of Proposition 17. The proof of the proposition follows the one of Theorem 2 in [13] (see also Theorem 1 in [22]) but, after Eq. (2.14), instead of applying the bound of Theorem 1 in [13], we apply the bound of our Proposition 15. Therefore, (2.15) in [13] becomes

$$\left| \int_{\mathbb{S}^r} f(\mathbf{x})\, \mathrm{d}\omega_r^\star(\mathbf{x}) - \frac{1}{N}\sum_{i=1}^{N} f(\mathbf{x}_i) \right| \leq \|f - f_m\|_{\mathcal{C}} + \max_{1\leq\ell\leq 2m} |A_\ell| \cdot \|f_m\|_{L_2} \cdot D_{(2m)}(\mathbf{P}_N; \{A_\ell\}) .$$

The rest of the proof follows the same lines, apart from the fact that we use $\|f - f_m\|_{\mathcal{C}} \leq \frac{\sqrt{20} r C_f}{m}$ (cf. [22, p. 331], where the authors majorize $\sqrt{20}$ through the integer 6). □

Proof of Proposition 19. We start from Theorem 1 in [2]. We identify $\mu(\cdot)$ with $\omega_r^\star(\cdot)$ and $\nu(\cdot)$ with the empirical measure induced by the configuration of points $\mathbf{P}_N$. Then, we majorize $C(m, \nu, r+1)$ (notice that we denote by $\mathbb{R}^{r+1}$ what the authors call $\mathbb{R}^d$) through Proposition 15. We identify $s = \left\lceil \frac{r+3}{2} \right\rceil$ and we majorize $\|f\|_{L_2} \leq \|f\|_{\mathcal{C}} \leq 1$. □

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