Computational Statistics and Data Analysis 71 (2014) 803–817

Contents lists available at ScienceDirect

Computational Statistics and Data Analysis

journal homepage: www.elsevier.com/locate/csda

Bootstrap confidence sets for the Aumann mean of a random closed set

Christine Choirat [a], Raffaello Seri [b,*]

[a] Department of Economics, School of Economics and Business Management, Universidad de Navarra, Edificio de Bibliotecas (Entrada Este), E-31080 Pamplona, Spain

[b] Dipartimento di Economia, Università dell'Insubria, Via Monte Generoso 71, I-21100 Varese, Italy

ARTICLE INFO

Article history:
Received 3 April 2012
Received in revised form 19 October 2012
Accepted 21 October 2012
Available online 17 November 2012

Keywords:
Random set
Aumann mean
Confidence set
Support function

ABSTRACT

The objective is to develop a reliable method to build confidence sets for the Aumann mean of a random closed set as estimated through the Minkowski empirical mean. First, a general definition of the confidence set for the mean of a random set is provided. Then, a method using a characterization of the confidence set through the support function is proposed and a bootstrap algorithm is described, whose performance is investigated in Monte Carlo simulations.

1. Introduction

After the first pioneering works of Matheron and Kendall (see Matheron, 1972; Kendall, 1974; Matheron, 1975), the study of random sets is receiving growing attention in the literature. Random sets have proved to be a valuable modelling tool in Economics, Physics and Biology and their theory offers a suitable framework to analyze old problems (as examples, consistency through Painlevé–Kuratowski convergence of epigraphs in statistics, see Choirat et al. (2003), transaction costs and risk measures in finance and partially identified models in econometrics, see Molchanov (2010)). However, statistical theory lags behind and most contributions deal with the properties of restricted classes of models, as the Boolean model (see, e.g., Molchanov, 1997). In this paper we aim at providing confidence sets for the mean of a random closed set (RACS). The literature on this topic is quite limited. In Seri and Choirat (2004), confidence sets based on Gaussian limit theory (see Theorem 1 below) are built for the Aumann mean of a RACS. In Jankowski and Stanberry (2012), the authors propose confidence sets for the Oriented Distance Function (ODF) mean (see Jankowski and Stanberry, 2010). The case of confidence sets for the mean of a fuzzy random variable is dealt with in González-Rodríguez et al. (2009). Further references on statistical procedures for the mean of fuzzy random variables are Montenegro et al. (2004), Gil et al. (2006), González-Rodríguez et al. (2012) and Ramos-Guajardo and Lubiano (2012).

After a review of the limit theory of RACSs in Euclidean spaces (Section 2), we provide some definitions of confidence sets for the mean of a RACS or, more properly, for the outline of its mean, since they contain it with prescribed probability (Section 3). One of the simplest ways — and the one we advocate here — to implement the confidence set is to use parallel bodies (see Schneider, 1993, pp. 134–135) and to resort to the support function embedding (see Schneider, 1993, Theorem 1.7.1). This yields asymptotically conservative/exact confidence regions that can be estimated through a bootstrap procedure. Computational details are dealt with in Section 4. In particular, in this section, using an example

* Corresponding author. Tel.: +39 0332 395515.
E-mail addresses: cchoirat@unav.es (C. Choirat), raffaello.seri@uninsubria.it (R. Seri).

0167-9473/$ – see front matter © 2012 Elsevier B.V. All rights reserved.
doi:10.1016/j.csda.2012.10.015

in which complete closed-form solutions exist, we discuss the dependence of the confidence set on the choice of the set relatively to which the parallel bodies are computed, then we compute the confidence sets on real data using the by-now classical sand grains data of Stoyan (1997), and at last we provide some Monte Carlo evidence about the performance of the method.

2. Limit theorems for RACSs

In the following, we introduce a number of definitions and we recall the Law of Large Numbers (LLN) and the Central Limit Theorem (CLT) for RACSs.

Consider the Euclidean space $\mathbb{R}^d$ endowed with the distance ρ, the Euclidean norm $\|\cdot\|$ and the inner product $\langle\cdot,\cdot\rangle$. Let $\mathbb{S}^{d-1} \triangleq \left\{u \in \mathbb{R}^d : \|u\| = 1\right\}$ be the hypersphere and $\mathsf{B} \triangleq \left\{u \in \mathbb{R}^d : \|u\| \leq 1\right\}$ be the ball, both with radius 1. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a complete probability space.

The *distance function* from a point x to the set $C \subset \mathbb{R}^d$ is:

$$\rho(x, C) \triangleq \inf_{y \in C} \rho(x, y).$$

The *norm* of a set C is $\|C\| \triangleq \sup\{\|x\|, x \in C\}$. The *support function* of C (see Ghosh and Kumar, 1998 for several useful properties and examples) is:

$$h(y, C) = h_C(y) \triangleq \sup_{x \in C} \langle y, x\rangle = \sup_{x \in C} \left(\sum_{i=1}^{d} y_i x_i\right).$$

To simplify the notation, whenever clear from the context we will remove the indication of the variable y from the support function. Since h is positively homogeneous of degree 1, it is customary to consider its restriction to $\mathbb{S}^{d-1}$. Therefore, in the following $h_C(\cdot)$ will be defined on $\mathbb{S}^{d-1}$. This function characterizes completely closed convex sets. Any subadditive real-valued function on $\mathbb{S}^{d-1}$ is the support function of a nonempty compact convex set (Theorem 1.7.1 in Schneider, 1993). The *Hausdorff distance* between two sets C and C' is defined by

$$\rho_H\left(C, C'\right) \triangleq \max\left\{\sup_{x \in C} \rho(x, C), \sup_{x' \in C'} \rho\left(x', C\right)\right\}.$$

Hörmander's formula relates the Hausdorff distance between sets to the L_∞-distance between support functions:

$$\rho_H\left(C, C'\right) = \sup_{y \in \mathbb{S}^{d-1}} |h_C(y) - h_{C'}(y)|$$

(see also Molchanov, 2005, p. 198, Eq. (1.8), or Schneider, 1993, p. 53, Theorem 1.8.11).

The *Minkowski sum* of two sets A and B is defined by:

$$A \oplus B \triangleq \{x + y : x \in A, y \in B\}.$$

The *Minkowski difference* of the sets A and B is:

$$A \ominus B \triangleq \{x : \{x\} \oplus B \subseteq A\}$$

(see, e.g., Schneider, 1993, p. 133, where a different symbol is used; see the same source, pp. 127 and 137 for other definitions of the difference of sets). In particular, this should not be confused with the *set-theoretic difference* of A and B or *relative complement* of B in A:

$$A \setminus B \triangleq \{x : x \in A, x \notin B\}.$$

The product of a scalar ε and a set A is:

$$\varepsilon A \triangleq \{\varepsilon \cdot x : x \in A\}.$$

If A and B are closed convex sets and $\varepsilon > 0$, we have:

$$h_{A \oplus B}(\cdot) = h_A(\cdot) + h_B(\cdot),$$
$$h_{A \ominus B}(\cdot) \leq h_A(\cdot) - h_B(\cdot),$$
$$h_{\varepsilon A}(\cdot) = \varepsilon h_A(\cdot)$$

(see, e.g., Schneider, 1993, respectively at p. 41, p. 134, p. 38). Let A, B be closed convex sets and let $\varepsilon \geq 0$ be a real number. The sets $A \oplus \varepsilon B$ and $A \ominus \varepsilon B$ are called *parallel bodies of A relative to B* (see Schneider, 1993, pp. 134–135). In particular, $A \oplus \varepsilon B$ is called outer parallel body and $A \ominus \varepsilon B$ is called inner parallel body.

Let $\mathfrak{F}$ ($\mathfrak{K}$) be the set of closed (compact) subsets of $\mathbb{R}^d$. A mapping $X : \Omega \to \mathfrak{F}$ is called a *random closed set* (RACS) if, for every $K \in \mathfrak{K}$, $\{\omega : X(\omega) \cap K \neq \varnothing\} \in \mathfrak{F}$ (see Molchanov, 2005, p. 1, Definition 1.1). A random closed set X with almost surely compact values (i.e. such that $X \in \mathfrak{K}\ \mathbb{P} - \text{as}$) is called a *random compact set* (see Molchanov, 2005, p. 21, Definition 1.30). A *measurable selection* from X is a measurable function $f : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R}^d$ that satisfies $f(\omega) \in X(\omega)$ for almost any $\omega \in \Omega$. The *Aumann mean* of a random closed set X is the set of integrals of measurable selections from X:

$$\mathbb{E}X = \left\{\mathbb{E}f : f \in L^1(\Omega, \mathcal{A}, \mathbb{P}), f(\omega) \in X(\omega)\ \mathbb{P} - \text{as}\right\}$$

(see Molchanov, 2005, p. 151, Definition 1.13). The Aumann mean of a RACS can be characterized through the support function as the set $\mathbb{E}X$ such that the following equality holds:

$$h_{\mathbb{E}X}(\cdot) = \mathbb{E}h_X(\cdot).$$

This characterization holds under conditions detailed in Molchanov (2005, Theorem 1.22, p. 157).

Now, we state some preliminary results. Let $(X_1, X_2, \ldots)$ be a sequence of iid random sets in $\mathbb{R}^d$. We define the Minkowski mean of the sample as:

$$\bar{X}_n \triangleq \frac{1}{n}\{X_1 \oplus X_2 \oplus \cdots \oplus X_{n-1} \oplus X_n\},$$

also indicated in the following as $\bar{X}_n = \frac{1}{n}\bigoplus_{i=1}^n X_i$. It is clear that $h_{\bar{X}_n} = \frac{1}{n}\sum_{i=1}^n h_{X_i}$. A well known result (see Artstein and Vitale, 1975; Puri and Ralescu, 1985; Artstein and Hansen, 1985) states that RACSs satisfy a LLN.

Theorem 1. *Let $X_1, X_2, \ldots$ be a sequence of iid random sets in $\mathbb{R}^d$ with $\mathbb{E}\|X_1\| < \infty$. Then*

$$\bar{X}_n \xrightarrow{\text{as}} \mathbb{E}X,$$

where convergence holds in the Hausdorff distance.

The Aumann mean is always a convex set, even if the random set X is not convex. However, this is not a problem, since Shapley–Folkman's inequality (see, e.g, Weil, 1982, p. 205) implies that the Hausdorff distance between $\frac{1}{n}\bigoplus_{i=1}^n X_i$ and $\frac{1}{n}\bigoplus_{i=1}^n \text{co}X_i$ (where co X_i is the convex hull of X_i) goes to 0 as $n \to \infty$.

CLTs for RACSs have been proved by Cressie (1979), Weil (1982), Ljašenko (1979), Giné et al. (1983) and Puri and Ralescu (1985). The paper (Cressie, 1979) first stated the CLT as an asymptotic distributional result for the Hausdorff distance between the empirical Minkowski mean and the Aumann mean, that is $\rho_H\left(\bar{X}_n, \mathbb{E}X\right)$, a formulation that has been adopted by most papers in the following. However, as we will show in the next section, we prefer to state our CLT in terms of support functions instead of Hausdorff distances.

Therefore, we get the following result (see Weil, 1982).

Theorem 2. *Let $X_1, X_2, \ldots$ be a sequence of iid random sets in $\mathbb{R}^d$ with $\mathbb{E}\|X_1\|^2 < \infty$. Then*

$$\sqrt{n}\cdot\left(h_{\bar{X}_n}(\cdot) - h_{\mathbb{E}X}(\cdot)\right) \xrightarrow{\mathcal{D}} Z(\cdot),$$

where $Z(\cdot)$ is a Gaussian centered process on $\mathbb{S}^{d-1}$ of covariance function $\Gamma_X(y,z) \triangleq \mathbb{E}(Z(y)Z(z))$ for $y, z \in \mathbb{S}^{d-1}$.

From Hörmander's formula, Weil (1982) obtains the following limit theorem for the Hausdorff distance between the empirical Minkowski mean of a sample of iid RACSs and its Aumann mean (see also Molchanov, 2005, p. 214, Theorem 2.1).

Corollary 3. *Let $X_1, X_2, \ldots$ be a sequence of iid random sets in $\mathbb{R}^d$ with $\mathbb{E}\|X_1\|^2 < \infty$. Then*

$$\sqrt{n}\cdot\rho_H\left(\bar{X}_n, \mathbb{E}X\right) \xrightarrow{\mathcal{D}} \sup_{y\in\mathbb{S}^{d-1}}|Z(y)|,$$

where $Z(\cdot)$ is a Gaussian centered process on $\mathbb{S}^{d-1}$ of covariance function $\Gamma_X(y,z) = \mathbb{E}(Z(y)Z(z))$ for $y, z \in \mathbb{S}^{d-1}$.

3. Theoretical aspects of confidence sets for the Aumann mean of a RACS

3.1. Definitions

Here is our definition of confidence sets for the mean of a random closed set. Even though our main interest is for the Aumann mean, this definition may be of broader interest.

In the following, we will use the notations $\simeq$ and $\gtrsim$. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of constants. In general, we write $a_n \simeq b_n$ (resp., $a_n \gtrsim b_n$) to say that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$ (resp., $\lim_{n\to\infty} a_n \geq \lim_{n\to\infty} b_n$). If b_n is independent of n, the definition varies accordingly.

Definition 4. Consider two sequences of random sets $\{K'_n\}$ and $\{K''_n\}$, with K'_n and K''_n constructed as Borel measurable functions of the sample $\{X_1, \ldots, X_n\}$. Suppose moreover that $K'_n \subseteq K''_n$ for every n. Then, the relative complement $K''_n \setminus K'_n$ is a confidence set of level α for the (outline of the) Aumann mean $\mathbb{E}X$ of the random closed set X if:

$$\mathbb{P}\left\{K'_n \subseteq \mathbb{E}X \subseteq K''_n\right\} \gtrsim 1-\alpha.$$

The confidence set is respectively called exact, conservative, asymptotically exact and asymptotically conservative if:

$$\begin{aligned}
&\mathbb{P}\left\{K'_n \subseteq \mathbb{E}X \subseteq K''_n\right\} = 1-\alpha,\\
&\mathbb{P}\left\{K'_n \subseteq \mathbb{E}X \subseteq K''_n\right\} \geq 1-\alpha,\\
&\lim_{n\to\infty} \mathbb{P}\left\{K'_n \subseteq \mathbb{E}X \subseteq K''_n\right\} = 1-\alpha,\\
&\lim_{n\to\infty} \mathbb{P}\left\{K'_n \subseteq \mathbb{E}X \subseteq K''_n\right\} \geq 1-\alpha.
\end{aligned}$$

If $K'_n = \varnothing$, we call K''_n an external confidence set. If K''_n coincides with the whole space, we call K'_n an internal confidence set.

In this paper, we will pursue the idea of using an outer (inner) parallel body as an external (internal) confidence set.

Definition 5. Confidence sets based on parallel bodies are defined as $K'_n \triangleq \bar{X}_n \ominus \lambda'_n B$ and $K''_n \triangleq \bar{X}_n \oplus \lambda''_n B$, where λ'_n and λ''_n are positive constants chosen to respect one of the formulas in Definition 4. The set B is a convex structuring element with nonempty interior and properly containing the origin.

The set B allows the researcher to control the extent of the confidence sets in all the directions. As an example, in the case of isotropic random closed sets, i.e. random sets whose distribution is invariant under rotations around the origin (see Molchanov, 2005, p. 49), it is natural to take $B = \mathsf{B}$, the unit ball; this choice has also definite computational advantages since $h_{\mathsf{B}}(\cdot) \equiv 1$, and this simplifies the computations. But in some cases, a different choice can lead to better confidence regions, as shown in Section 4.1. The fact that B is convex, with nonempty interior and properly contains the origin implies that $h_B(\cdot) > 0$ for every y, a property that will be used in the following. Remark that it would be possible to consider two different structuring elements for the internal and the external confidence sets: however, we leave the necessary changes to the reader.

3.2. Confidence sets through support functions

Using the support function embedding, we get:

$$\begin{aligned}
\mathbb{P}\left\{K'_n \subseteq \mathbb{E}X \subseteq K''_n\right\} &= \mathbb{P}\left\{h_{K'_n} \leq h_{\mathbb{E}X} \leq h_{K''_n}\right\}\\
&= \mathbb{P}\left\{h_{\bar{X}_n \ominus \lambda'_n B} \leq h_{\mathbb{E}X} \leq h_{\bar{X}_n \oplus \lambda''_n B}\right\}\\
&= \mathbb{P}\left\{h_{\bar{X}_n \ominus \lambda'_n B} \leq h_{\mathbb{E}X} \leq h_{\bar{X}_n} + \lambda''_n \cdot h_B\right\}\\
&\geq \mathbb{P}\left\{h_{\bar{X}_n} - \lambda'_n \cdot h_B \leq h_{\mathbb{E}X} \leq h_{\bar{X}_n} + \lambda''_n \cdot h_B\right\}\\
&= \mathbb{P}\left\{\lambda'_n \geq \frac{h_{\bar{X}_n} - h_{\mathbb{E}X}}{h_B} \geq -\lambda''_n\right\},
\end{aligned} \tag{3.1}$$

where the fourth step comes from the relation $h_{\bar{X}_n \ominus \lambda'_n B} \leq h_{\bar{X}_n} - \lambda'_n \cdot h_B$ (see, e.g., Schneider, 1993, p. 134).

It is clear that the support function embedding allows one to write:

$$\begin{aligned}
\mathbb{P}\left\{K'_n \subseteq \mathbb{E}X \subseteq K''_n\right\} &\geq \mathbb{P}\left\{\left(\sqrt{n}\lambda'_n\right) \cdot h_B \geq \sqrt{n}\left(h_{\bar{X}_n} - h_{\mathbb{E}X}\right) \geq -\left(\sqrt{n}\lambda''_n\right) \cdot h_B\right\}\\
&\simeq \mathbb{P}\left\{\left(\sqrt{n}\lambda'_n\right) \cdot h_B \geq Z \geq -\left(\sqrt{n}\lambda''_n\right) \cdot h_B\right\},
\end{aligned}$$

where Z is the process of Theorem 2, and $\sqrt{n}\lambda'_n$ or $\sqrt{n}\lambda''_n$ should be chosen accordingly. This strategy has been pursued by Seri and Choirat (2004) in a more limited context, but has some drawbacks. In particular, the method works as follows. First, the process Z is replaced by a discrete version supported by a subset of p points on the sphere $\mathbb{S}^{d-1}$; this, in particular, implies that the covariance function Γ_X of Z is replaced by a covariance matrix. Second, this covariance matrix is estimated using the values of the support functions in the p points. The first drawback is that the discretization introduces an error that decreases when $p \to \infty$ but is difficult to control. This is worsened by the second drawback of the method, namely the fact that, in order to obtain a nonsingular covariance matrix, p has to be smaller than n. The second drawback could be corrected using one of the methods available to estimate covariance matrices with $p > n$. However, even in this case, we do not expect the method to work better than the one proposed in the present paper, that is based on the bootstrap.

It is helpful to remark that (3.1) leads to:

$$\begin{aligned}\mathbb{P}\left\{\lambda_n' \geq \frac{h_{\bar{X}_n} - h_{\mathbb{E}X}}{h_B} \geq -\lambda_n''\right\} &= \mathbb{P}\left(\left\{\lambda_n' \geq \sup_y \frac{h_{\bar{X}_n} - h_{\mathbb{E}X}}{h_B}\right\} \bigcap \left\{\inf_y \frac{h_{\bar{X}_n} - h_{\mathbb{E}X}}{h_B} \geq -\lambda_n''\right\}\right) \\ &= 1 - \mathbb{P}\left\{\lambda_n' < \sup_y \frac{h_{\bar{X}_n} - h_{\mathbb{E}X}}{h_B}\right\} - \mathbb{P}\left\{\inf_y \frac{h_{\bar{X}_n} - h_{\mathbb{E}X}}{h_B} < -\lambda_n''\right\} \\ &\quad + \mathbb{P}\left(\left\{\lambda_n' < \sup_y \frac{h_{\bar{X}_n} - h_{\mathbb{E}X}}{h_B}\right\} \bigcap \left\{\inf_y \frac{h_{\bar{X}_n} - h_{\mathbb{E}X}}{h_B} < -\lambda_n''\right\}\right) \\ &\geq 1 - \mathbb{P}\left\{\lambda_n' < \sup_y \frac{h_{\bar{X}_n} - h_{\mathbb{E}X}}{h_B}\right\} - \mathbb{P}\left\{\inf_y \frac{h_{\bar{X}_n} - h_{\mathbb{E}X}}{h_B} < -\lambda_n''\right\}.\end{aligned}$$

Choosing λ_n' and λ_n'' in such a way that the following equalities hold asymptotically:

$$\begin{aligned}&\mathbb{P}\left\{\lambda_n' < \sup_y \left(\frac{h_{\bar{X}_n} - h_{\mathbb{E}X}}{h_B}\right)\right\} \simeq \alpha_1, \\ &\mathbb{P}\left\{\inf_y \left(\frac{h_{\bar{X}_n} - h_{\mathbb{E}X}}{h_B}\right) < -\lambda_n''\right\} \simeq \alpha_2,\end{aligned}$$

where α_1 and α_2 are such that $\alpha_1 + \alpha_2 = \alpha$, we get a confidence set with asymptotic coverage at least $1 - \alpha$.

The choice of λ_n' and λ_n'' (or of α_1 and α_2) may follow five methods (see Hall and Pittelkow, 1990, for analogues of the last three in the case of confidence bands in regression):

1. One-sided external regions, in which the Aumann mean is expected to be included with prescribed probability: this means that $K_n' = \varnothing$, $\alpha_1 = 0$ or $\lambda_n' = +\infty$. Remark that the equality $\lambda_n' = +\infty$ is only a formal way to state that $K_n' = \varnothing$, since λ_n' equal to the inradius of $\bar{X}_n$ relative to B would be enough (see Schneider, 1993, p. 135). We expect these sets to be asymptotically exact, since the inequality above becomes an equality. In this case, we can write:
$$\begin{aligned}&1 - \alpha \simeq \mathbb{P}\left\{\inf_y \left(\frac{h_{\bar{X}_n} - h_{\mathbb{E}X}}{h_B}\right) \geq -\lambda_n''\right\} = \mathbb{P}\left\{\mathbb{E}X \subseteq K_n''\right\}, \\ &\alpha \simeq \mathbb{P}\left\{\inf_y \left(\frac{h_{\bar{X}_n} - h_{\mathbb{E}X}}{h_B}\right) < -\lambda_n''\right\}.\end{aligned}$$
2. One-sided internal regions, that are contained in the Aumann mean with prescribed probability: this means that $K_n'' = \mathbb{R}^d$, $\alpha_2 = 0$ or $\lambda_n'' = +\infty$. In this case:
$$1 - \alpha \simeq \mathbb{P}\left\{\sup_y \left(\frac{h_{\bar{X}_n} - h_{\mathbb{E}X}}{h_B}\right) \leq \lambda_n'\right\} \leq \mathbb{P}\left\{K_n' \subseteq \mathbb{E}X\right\}.$$
We expect this set to be asymptotically exact only in special cases (in this respect, the concept of summand, see Schneider, 1993, p. 134, can turn out to be useful).
3. Two-sided symmetric regions: the internal and external sets are determined in such a way that the two quantiles are equal, i.e. $\lambda_n' = \lambda_n''$:
$$\mathbb{P}\left\{\sup_y \left|\frac{h_{\bar{X}_n} - h_{\mathbb{E}X}}{h_B}\right| \leq \lambda_n'\right\} \simeq 1 - \alpha.$$
Remark that, when $B = \mathsf{B}$, this corresponds to the Weil's CLT seen above.
4. Two-sided equal-tailed regions: the probability of the outline lying in the region is predetermined, and the probability of the external set containing and of the internal set being contained in the Aumann mean determined in this way are (almost) equal, i.e. $\alpha_1 = \alpha_2$. Therefore:
$$\mathbb{P}\left\{\lambda_n' < \sup_y \left(\frac{h_{\bar{X}_n} - h_{\mathbb{E}X}}{h_B}\right)\right\} \simeq \mathbb{P}\left\{\inf_y \left(\frac{h_{\bar{X}_n} - h_{\mathbb{E}X}}{h_B}\right) < -\lambda_n''\right\} \simeq \frac{\alpha}{2}.$$
5. Two-sided narrowest-width regions: in this case the sets are determined as to minimize the width of the region, i.e. the sum of the two quantiles. The problem becomes:
$$\min_{\lambda_n', \lambda_n''} \lambda_n' + \lambda_n''$$
$$\text{under} \begin{cases} \mathbb{P}\left\{\lambda_n' < \sup_y \left(\frac{h_{\bar{X}_n} - h_{\mathbb{E}X}}{h_B}\right)\right\} \simeq \alpha_1, \\ \mathbb{P}\left\{\inf_y \left(\frac{h_{\bar{X}_n} - h_{\mathbb{E}X}}{h_B}\right) < -\lambda_n''\right\} \simeq \alpha_2, \\ \alpha_1 + \alpha_2 = \alpha. \end{cases}$$

Table 1
Bootstrap formulas for quantiles of confidence sets.

		Bootstrap
One-sided	External	$\alpha = \mathbb{P}\left\{\inf_y \left(\frac{h^\star_{\bar{X}_n} - h_{\bar{X}_n}}{h_B}\right) < -\lambda''_n \,\middle|\, \mathcal{P}_n\right\}$
	Internal	$1-\alpha = \mathbb{P}\left\{\sup_y \left(\frac{h^\star_{\bar{X}_n} - h_{\bar{X}_n}}{h_B}\right) \le \lambda'_n \,\middle|\, \mathcal{P}_n\right\}$
Two-sided	Symmetric	$\mathbb{P}\left\{\sup_y \left\|\frac{h^\star_{\bar{X}_n} - h_{\bar{X}_n}}{h_B}\right\| \le \lambda'_n \,\middle|\, \mathcal{P}_n\right\} = 1-\alpha$
	Equal-tailed	$\begin{cases}\mathbb{P}\left\{\sup_y \left(\frac{h^\star_{\bar{X}_n} - h_{\bar{X}_n}}{h_B}\right) \le \lambda'_n \,\middle|\, \mathcal{P}_n\right\} = 1-\frac{\alpha}{2}\\ \mathbb{P}\left\{\inf_y \left(\frac{h^\star_{\bar{X}_n} - h_{\bar{X}_n}}{h_B}\right) < -\lambda''_n \,\middle|\, \mathcal{P}_n\right\} = \frac{\alpha}{2}\end{cases}$
	Narrowest-width	$\begin{cases}\min_{\lambda'_n,\lambda''_n} \lambda'_n + \lambda''_n\\ \mathbb{P}\left\{\sup_y \left(\frac{h^\star_{\bar{X}_n} - h_{\bar{X}_n}}{h_B}\right) \le \lambda'_n \,\middle|\, \mathcal{P}_n\right\} = 1-\alpha_1\\ \mathbb{P}\left\{\inf_y \left(\frac{h^\star_{\bar{X}_n} - h_{\bar{X}_n}}{h_B}\right) < -\lambda''_n \,\middle|\, \mathcal{P}_n\right\} = \alpha_2\\ \alpha_1+\alpha_2 = \alpha\end{cases}$

3.3. The bootstrap algorithm

In this paper, we propose to use the bootstrap to approximate the quantiles λ'_n and λ''_n seen above. This is clearly a case of bootstrap for functional data (Cuevas et al., 2006; McMurry and Politis, 2011).

Let $\mathbb{P}_n$ be the *empirical distribution* of the sample $\mathcal{P}_n \triangleq \left\{h_{X_1}, h_{X_2}, \ldots, h_{X_n}\right\}$, defined as $\mathbb{P}_n = n^{-1}\sum_{i=1}^n \delta_{h_{X_i}}$, where δ_h is the Dirac mass in h. Suppose to draw with replacement a sample $\mathcal{P}^\star_n \triangleq \left\{h^\star_{X_1}, h^\star_{X_2}, \ldots, h^\star_{X_n}\right\}$ from $\mathcal{P}_n$. The bootstrap probability is defined as the empirical probability of $\mathcal{P}^\star_n$, conditionally on $\mathcal{P}_n$, namely as $\mathbb{P}^\star_n\,|\mathcal{P}_n = n^{-1}\sum_{i=1}^n \delta_{h^\star_{X_i}}$. Let T be a functional defined on a set of probabilities containing $\mathbb{P}$, $\mathbb{P}_n$ and $\mathbb{P}^\star_n$. As an example, in the present case, $T(\mathbb{P}) = \frac{\int_\Omega h_{X(\omega)}\mathrm{d}\mathbb{P}(\omega)}{h_B}$. Our interest is to study the distribution of the difference $[T(\mathbb{P}_n) - T(\mathbb{P})]$. It turns out that $\left[T\left(\mathbb{P}^\star_n\right) - T(\mathbb{P}_n)\right]|\mathcal{P}_n$ is a good approximation to this distribution, that in general works much better than usual asymptotic approximations. Remark that the only variability in $\left[T\left(\mathbb{P}^\star_n\right) - T(\mathbb{P}_n)\right]|\mathcal{P}_n$ comes from the sampling mechanism from the fixed empirical probability $\mathbb{P}_n$. This is evident if we write $\mathbb{P}^\star_n\,|\mathcal{P}_n = n^{-1}\sum_{i=1}^n M_{ni}\cdot\delta_{h_{X_i}}$ where M_{ni} is the (random) number of times that (the fixed) h_{X_i} appears in $\mathcal{P}^\star_n$, and the vector $\mathbf{M}_n = (M_{n1}, \ldots, M_{nn}) \sim \mathrm{Mult}\left(n; \frac{1}{n}, \ldots, \frac{1}{n}\right)$ is distributed according to a multinomial distribution with parameter n and probabilities $\left(n^{-1}, \ldots, n^{-1}\right)$.

Using the bootstrap, we replace the original quantities with their bootstrap analogues conditionally on the observed sample $\mathcal{P}_n$:

$$\mathbb{P}\left\{\lambda'_n \ge [T(\mathbb{P}_n) - T(\mathbb{P})] \ge -\lambda''_n\right\} \simeq \mathbb{P}\left\{\lambda'_n \ge \left[T\left(\mathbb{P}^\star_n\right) - T(\mathbb{P}_n)\right] \ge -\lambda''_n \,|\mathcal{P}_n\right\}$$

or:

$$\mathbb{P}\left\{\lambda'_n \ge \frac{h_{\bar{X}_n} - h_{\mathbb{E}X}}{h_B} \ge -\lambda''_n\right\} \simeq \mathbb{P}\left\{\lambda'_n \ge \frac{h^\star_{\bar{X}_n} - h_{\bar{X}_n}}{h_B} \ge -\lambda''_n \,|\mathcal{P}_n\right\}.$$

Following the same derivations that have led to the formulas for methods 1–5, we get the bootstrap formulas in Table 1.

We are left with the problem of verifying the theoretical convergence of the bootstrap confidence regions. This is the subject of the next theorem, whose proof is very simple. A similar result is Proposition 2.1 in Beresteanu and Molinari (2008), but Theorem 6 is more general since it provides a necessary and sufficient condition and more suitable for our aims since it is stated in terms of the support function.

Theorem 6. *Let $X_1, X_2, \ldots$ be a sequence of iid random sets in $\mathbb{R}^d$. Then the condition $\mathbb{E}\|X_1\|^2 < \infty$ is equivalent to:*

$$\sqrt{n}\cdot\left(h^\star_{\bar{X}_n}(\cdot) - h_{\bar{X}_n}(\cdot)\right) \xrightarrow{\mathcal{D}} Z(\cdot) \quad \text{as},$$

where $Z(y)$ is the Gaussian centered process defined in Theorem 2.

Proof. The functions h_{X_i}, for $i = 1, \ldots, n$, are iid $\mathcal{C}\left(\mathbb{S}^{d-1}\right)$-valued random elements, where $\mathcal{C}\left(\mathbb{S}^{d-1}\right)$ is a separable Banach space. Therefore, Remark 2.5 in Giné and Zinn (1990) or Corollary 2 in McMurry and Politis (2011) can be applied. The condition $\mathbb{E}\|X_1\|^2 < \infty$ guarantees that the CLT holds as in Theorem 2. This, together with $\mathbb{E}\left\|h_{X_1}\right\|^2 < \infty$, guarantees that the convergence result in the statement holds true. From $\mathbb{E}\left\|h_{X_1}\right\|^2 = \mathbb{E}\|X_1\|^2$ (see, e.g., Weil, 1982, p. 206), we get the result. □

A first problem, common to most instances of the bootstrap, is that the probability $\mathbb{P}_n^\star|\mathcal{P}_n$ is known but difficult to manage. Therefore, it is generally approximated randomly drawing a set of J new samples $\mathcal{P}_{nj}^\star$ for $j = 1, \ldots, J$ of size n, each one with replacement from the original one $\mathcal{P}_n$, and building the empirical cdf $\mathbb{P}_n^{\star J}|\mathcal{P}_n$. If J is large enough, $\left[T\left(\mathbb{P}_n^{\star,J}\right) - T\left(\mathbb{P}_n\right)\right]|\mathcal{P}_n \simeq \left[T\left(\mathbb{P}_n^\star\right) - T\left(\mathbb{P}_n\right)\right]|\mathcal{P}_n$.

A second problem is that, except for special cases, the functions h_X can be handled only after a discretization. This is usually done replacing the support function with its discretized version along p directions on the sphere $\mathbb{S}^{d-1}$. In the special case $d = 2$, that is the one we will consider in our simulations, it is possible to discretize h_X taking p equi-spaced points on $\mathbb{S}^1$. In all other cases, the choice of the directions can be helped using deterministic sequences defined on the sphere (see Cui and Freeden, 1997; Choirat and Seri, forthcoming, and references therein). In the simulations of Section 4.3, we will try to investigate empirically the effect of the discretization error letting p vary from 5 to 640.

4. Computational aspects of confidence sets for the Aumann mean of a RACS

4.1. The choice of the set B

We consider an example in $\mathbb{R}^2$ that has the advantage of offering a complete closed-form solution. Since our aim is just to show that the choice of the set B can influence the form of the confidence set, we will not use the bootstrap algorithm described above, but we will use exact quantiles; the extension to quantiles obtained through the asymptotic normal approximation or the bootstrap is left to the reader. We only deal explicitly with one-sided external and internal confidence sets based on exact quantiles, while two-sided confidence sets are briefly considered at the end of the section. Moreover, we first provide formulas for the exact confidence sets as obtained through geometric reasoning. After that, we provide a derivation of the confidence sets proposed above using the support function. The derivation of the expressions for λ_n' and λ_n'' is briefly outlined in Appendix A.

We consider a sample of sets defined as $X_i = E_i \cdot H$, where $E_i \sim \mathcal{E}(1)$, an exponential random variable with parameter 1, and $H \triangleq [-1, +1]^2$, the filled square of side 2 centered at the origin. We write $\bar{E}_n = \frac{1}{n}\sum_{i=1}^n E_i$, where $n \cdot \bar{E}_n \sim \Gamma(n, 1)$; let $\gamma_{n,\alpha}$ be the lower α-quantile of $\Gamma(n, 1)$. The simplest way to obtain h_{X_i} is to remark that only the vertices of X_i contribute to it (since a set and its convex hull have the same support function); then, using the definition of support function and some simple trigonometry, it can be shown that:

$$h_{X_i}(y) = \sqrt{2}E_i \cdot \max_{\theta \in \left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}} \cos(y - \theta).$$

From this, we get $h_{\mathbb{E}X}(y) = \sqrt{2} \cdot \max_{\theta \in \left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}} \cos(y - \theta)$, $h_{\bar{X}_n}(y) = \sqrt{2} \cdot \bar{E}_n \cdot \max_{\theta \in \left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}} \cos(y - \theta)$ and $\bar{X}_n = \bar{E}_n \cdot H$.

We start from the confidence sets based on geometric reasoning. Consider first the external confidence sets. If we take $B = \mathsf{B}$, we have $K_n'' = \bar{X}_n \oplus \sqrt{2} \cdot \left(1 - \frac{\gamma_{n,\alpha}}{n}\right)\mathsf{B}$; if we take $B = H$, we have $K_n'' = \bar{X}_n \oplus \left(1 - \frac{\gamma_{n,\alpha}}{n}\right)H$. Consider now the internal confidence sets. If we take $B = \mathsf{B}$, we have $K_n' = \bar{X}_n \ominus \left(\frac{\gamma_{n,1-\alpha}}{n} - 1\right)\mathsf{B}$; if we take $B = H$, we have $K_n' = \bar{X}_n \ominus \left(\frac{\gamma_{n,1-\alpha}}{n} - 1\right)H$. It is not completely clear from the formulas that indeed $\bar{X}_n \ominus \left(\frac{\gamma_{n,1-\alpha}}{n} - 1\right)\mathsf{B} = \bar{X}_n \ominus \left(\frac{\gamma_{n,1-\alpha}}{n} - 1\right)H$, so that the two sets yield exactly the same internal confidence region. This is not true for the external sets: in this case, the square leads to the smallest confidence set with the required coverage, while the ball leads to a set that is a little larger than what really needed, despite remaining exact.

As far as the construction based on the support function is concerned, consider first the external confidence sets. The one based on $B = \mathsf{B}$ offers no difficulty: using the approach of Section 3.2, the quantile turns out to be $\lambda_n'' = \sqrt{2}\left(1 - \frac{\gamma_{n,\alpha}}{n}\right)$ and the support function is $h_{\bar{X}_n \oplus \lambda_n''\mathsf{B}}(y) = \sqrt{2} \cdot \bar{E}_n \cdot \max_{\theta \in \left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}} \cos(y - \theta) + \lambda_n''$. Remark that the set thus obtained is a square with rounded corners: a smaller confidence set with the same coverage level can be obtained choosing $B = H$. In this case, the quantile is $\lambda_n'' = 1 - \frac{\gamma_{n,\alpha}}{n}$ and the external confidence set has support function given by $h_{\bar{X}_n \oplus \lambda_n'' H}(y) = \sqrt{2} \cdot \left(\bar{E}_n + \lambda_n''\right) \cdot \max_{\theta \in \left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}} \cos(y - \theta)$. This is a square and is the smallest possible set with coverage level equal to $1 - \alpha$. Remark that the external sets coincide with the ones obtained through geometric reasoning.

Now consider the construction based on the support functions for the internal sets. If we take $B = H$, the approach of Section 3.2 yields the quantile $\lambda_n' = \frac{\gamma_{n,1-\alpha}}{n} - 1$, the support function $h_{\bar{X}_n}(y) - h_{\lambda_n' H}(y) = h_{\bar{X}_n \ominus \lambda_n' H}(y) = \sqrt{2} \cdot \left(\bar{E}_n - \lambda_n'\right) \cdot \max_{\theta \in \left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}} \cos(y - \theta)$ and the set $K_n' = \bar{X}_n \ominus \left(\frac{\gamma_{n,1-\alpha}}{n} - 1\right)H$.

On the other hand, if we take $B = \mathsf{B}$, some problems arise. Here we get $\lambda'_n = \sqrt{2}\left(\frac{\gamma_{n,1-\alpha}}{n} - 1\right)$; the approach of Section 3.2 is based on the function $h_{\bar{X}_n}(y) - h_{\lambda'_n \mathsf{B}}(y) = \sqrt{2}\cdot\bar{E}_n \cdot \max_{\theta\in\left\{\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}\right\}} \cos(y-\theta) - \lambda'_n$, that is not a proper support function (since it fails to be subadditive). Inverting it as if it were a support function, we get the set $K'_n = \bar{X}_n \ominus \sqrt{2}\left(\frac{\gamma_{n,1-\alpha}}{n} - 1\right)\mathsf{B} = \bar{X}_n \ominus \sqrt{2}\left(\frac{\gamma_{n,1-\alpha}}{n} - 1\right)H$ (whose support function is $h_{\bar{X}_n \ominus \lambda'_n \mathsf{B}}(y) = \sqrt{2}\cdot\left(\bar{E}_n - \lambda'_n\right)\cdot \max_{\theta\in\left\{\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}\right\}} \cos(y-\theta)$). Using the formulas for λ'_n and λ''_n introduced above in the support function case, it is simple to see that the internal set based on $B = \mathsf{B}$ is smaller than the one based on $B = H$.

Figs. 4.1 and 4.2 provide a graphical illustration in the spirit of Ghosh and Kumar (1998) of the confidence sets based on the support functions. To simplify, we take $n = 10$ and we suppose that $\bar{X}_n = H$: the choice of $n = 10$ is due to the fact that larger values of n yield smaller confidence sets that would not be easy to distinguish in the figures; the choice of $\bar{X}_n = H$ is simply a convenient one, but it is clear that in applications $\bar{X}_n$ would be the Minkowski mean of a sample of sets. Fig. 4.1 provides an illustration of the two confidence sets based on the ball $B = \mathsf{B}$. The quantities λ'_n and λ''_n are given, as above, by $\lambda'_n = \sqrt{2}\cdot\left(\frac{\gamma_{n,1-\alpha}}{n} - 1\right)$ and $\lambda''_n = \sqrt{2}\cdot\left(1 - \frac{\gamma_{n,\alpha}}{n}\right)$. In the upper (lower) left quadrant of the graph, we represent the set $\lambda''_n B = \lambda''_n \mathsf{B}$ ($\lambda'_n B = \lambda'_n \mathsf{B}$) centered at $(1, 1)$ as a grey filled ball, the outline of the set $\bar{X}_n = H$ as a solid black line, the outline of the 95% external (internal) confidence set $\bar{X}_n \oplus \lambda''_n \mathsf{B}$ as a dashed black line and the outline of the largest (smallest) possible $\mathbb{E}X$ contained in (containing) the confidence set as a solid thin grey line. In the upper (lower) right quadrant we represent the support function $h_{\bar{X}_n}$ of the set $\bar{X}_n = H$ as a solid black line, the support function $h_{\bar{X}_n \oplus \lambda''_n \mathsf{B}}$ ($h_{\bar{X}_n \ominus \lambda'_n \mathsf{B}}$) of the 95% external (internal) confidence set $\bar{X}_n \oplus \lambda''_n \mathsf{B}$ ($\bar{X}_n \ominus \lambda'_n \mathsf{B}$) as a dashed black line, and the support function of the largest (smallest) possible $\mathbb{E}X$ contained in (containing) the confidence set as a solid thin grey line; moreover, in the lower quadrant, we also represent the function $h_{\bar{X}_n} - \lambda'_n$ (that is not a proper support function) as a dashed grey line. Fig. 4.2 represents the same quantities when $B = H$. As above, the quantities λ'_n and λ''_n are given by $\lambda'_n = \frac{\gamma_{n,1-\alpha}}{n} - 1$ and $\lambda''_n = 1 - \frac{\gamma_{n,\alpha}}{n}$. Remark that here there is no difference between the external (internal) confidence set and the largest (smallest) possible $\mathbb{E}X$ contained in (containing) it.

Now we outline the derivation of the quantiles in the two-sided case. We only consider the construction based on the support functions, since this is the one that is used in practice. The idea is to choose λ'_n (λ''_n) as the quantile of an internal (external) confidence set at level α_1 (α_2); here, $\alpha_1 + \alpha_2 = \alpha$ and the choice of α_1 and α_2 depends on the method. In the following, we will see that the choice of α_1 and α_2 for the different methods does not depend on whether $B = H$ or $B = \mathsf{B}$, but this does not seem to hold in general. For the two-sided symmetric case, we choose $\lambda'_n = \lambda''_n$ or $\gamma_{n,1-\alpha_1} + \gamma_{n,\alpha_2} = 2n$ with $\alpha_1 + \alpha_2 = \alpha$. In the two-sided equal-tailed case, we take $\alpha_1 = \alpha_2 = \frac{\alpha}{2}$ and we simply choose λ'_n and λ''_n as the quantiles of the corresponding internal and external confidence sets at level $\frac{\alpha}{2}$. At last, for the two-sided narrowest-width confidence set, we choose α_1 and α_2 to minimize $\lambda'_n + \lambda''_n$; both for $B = H$ and for $B = \mathsf{B}$, this is equivalent to minimize $\gamma_{n,1-\alpha_1} - \gamma_{n,\alpha_2}$ under the constraint $\alpha_1 + \alpha_2 = \alpha$.

4.2. An example on real data

In this section, we compute our confidence sets on the sand grains data first analyzed in Stoyan (1997). A set of 25 sand grains comes from the banks of the Zelenchuk River in Ossetia; the other set comes from the shores of the Baltic Sea and is composed of 24 elements. The particles were photographed maintaining the same scale; the result is composed of two-dimensional projections represented by binary images. The resulting images are rounded but not necessarily convex. River grains are more elongated and smaller, sea grains appear to be more spherical and larger.

The outlines of the sand grains were represented as 50 vertices at approximately equal arc-length, as described at the end of Section 8.1 in Kent et al. (2000). Before computing the confidence sets, the images have been realigned and scaled using the generalized Procrustes analysis as implemented in the shapes package (Dryden, 2012) in R (see R Development Core Team, 2011).

Fig. 4.3 shows the confidence regions at 90% based on $J = 10{,}000$ bootstrap samples obtained using $B = \mathsf{B}$. We chose to represent them at 90%, since at 95% they would be too narrow to be distinguished from the outline of the Minkowski mean. The first two rows concern sand grains respectively from the Zelenchuk river and the Baltic sea that have been aligned without scaling; the last two rows concern the same grains that have been aligned and scaled. The columns contain from left to right the one-sided external, the one-sided internal, the two-sided symmetric, the two-sided equal-tailed and the two-sided narrowest-width confidence sets. It is apparent that sets obtained on scaled sand grains are narrower than sets obtained without scaling, since the variability of scaled and aligned sand grains is smaller. The solid black line represents the Minkowski mean, the dashed lines represent the boundaries of the confidence regions.

4.3. Simulations

We have performed several Monte Carlo simulations. In this section, we only present graphs for a set of simulations; the graphs for the other set and the tables with the numerical results are in the Supplementary Material.

Let X be the convex hull of 5 points in $\mathbb{R}^2$ ($d = 2$) drawn from a bivariate standard normal distribution. $\mathbb{E}X$ is given by $1.1629644736 \cdot \mathsf{B}$, the ball with radius 1.1629644736. A proof of this fact can be found in Appendix B. We take

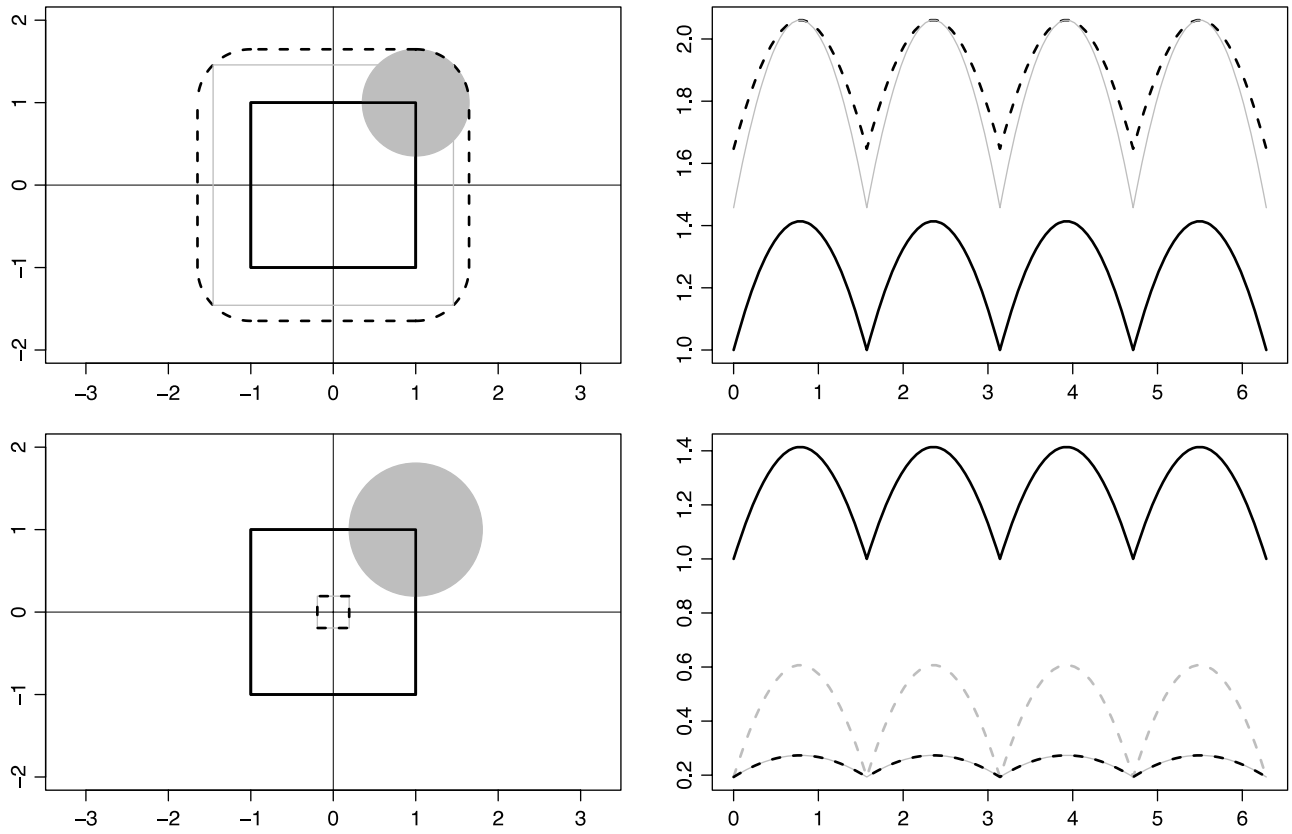

Fig. 4.1. Confidence sets constructed with $B = \mathrm{B}$.

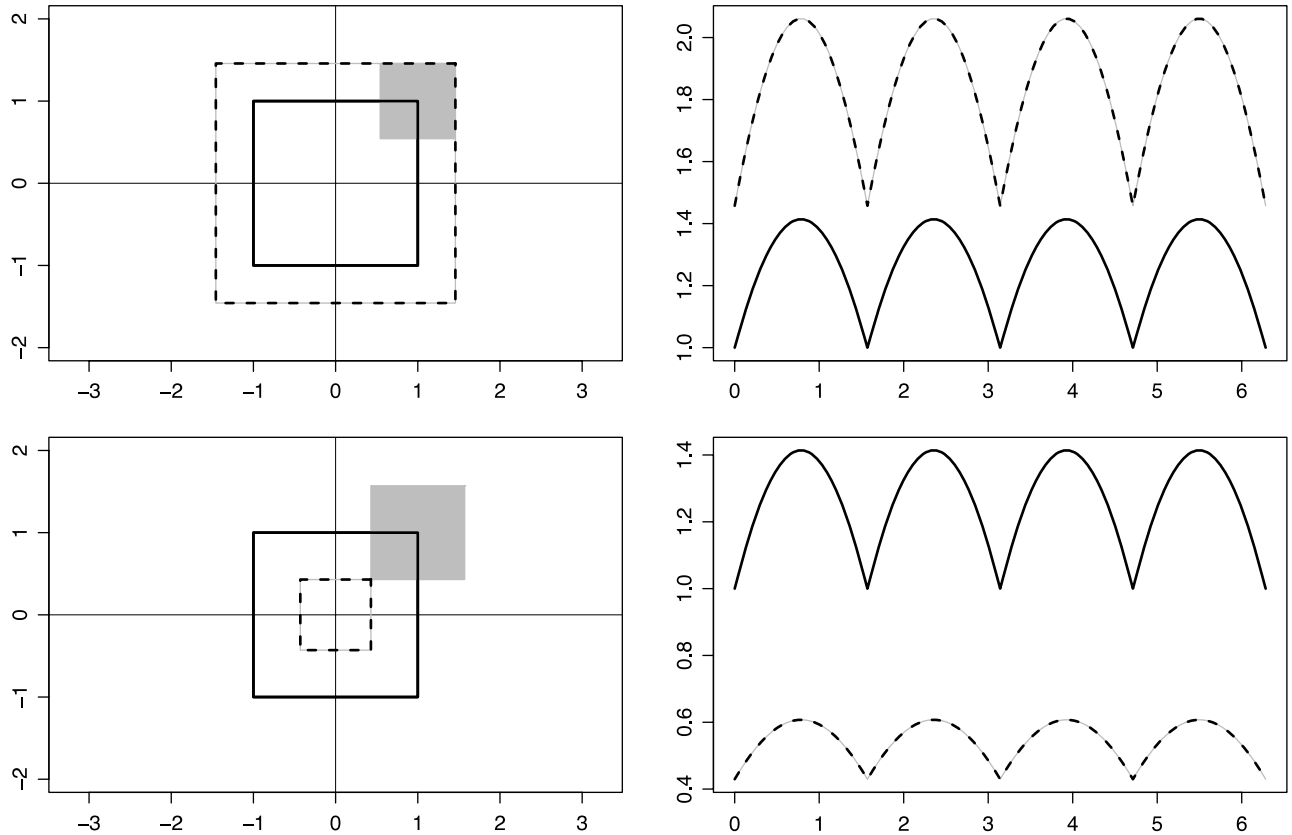

Fig. 4.2. Confidence sets constructed with $B = H$.

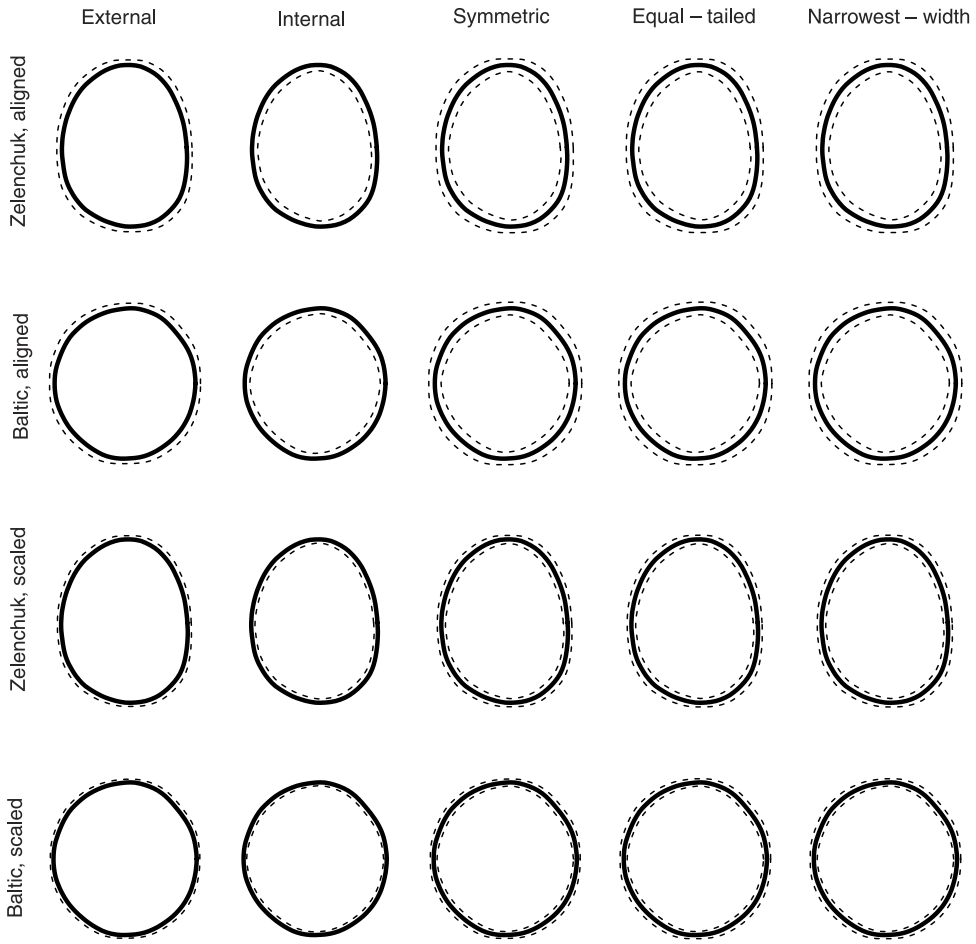

Fig. 4.3. Confidence sets for the sand grains data.

$n \in \{5, 10, 20, 40, 80, 160, 320, 640\}$ and $p \in \{5, 10, 20, 40, 80, 160, 320, 640\}$. For each sample, we compute λ_n' and λ_n'' using 1000 boostrap samples. The probability that $h_{\mathbb{E}X}$ is inside the confidence region is computed using $J = 10{,}000$ replications. Figs. 4.4–4.8 display the behavior of the coverage probability when n, p and α vary.

These are the main results:

1. As expected, low values of p lead to confidence bands with low coverage. This is due to the fact that, when p is small, λ_n' and λ_n'' are biased towards zero and therefore the confidence sets tend to be smaller and more permissive.
2. Low values of n lead to confidence bands with low coverage. The same phenomenon is observed in Jankowski and Stanberry (2012).
3. As expected, one-sided internal sets tend to be slightly more conservative than one-sided external sets.
4. For $\alpha = 0.1, 0.05$, two-sided symmetric and equal-tailed sets are quite precise, while narrowest-width sets are more permissive: this is due to the fact that $\lambda_n' + \lambda_n''$ is smaller for narrowest-width sets, that tend therefore to be smaller and less conservative.
5. All one-sided and two-sided sets with $\alpha = 0.01$ or 0.001 are less precise and more permissive than the ones with high values of α, also for large n and p. This may probably be reduced increasing the number J of bootstrap resamplings, fixed at 1000 in this simulation.
6. When the set X is far from a ball, a smaller value of p may be needed to reach the desired precision. Indeed, if $y_1, \ldots, y_p$ are the p discretization directions of h_X on $\mathbb{S}^{d-1}$, let $x_1^\star, \ldots, x_p^\star$ be the points for which $h_X\left(y_j\right) = \left\langle y_j, x_j^\star \right\rangle$. The fact that these points tend to cluster around the exposed points of X_i (see Schneider, 1993, p. 19) allows for a better description of the set X_i exactly where the curvature of its outline is larger.
7. Despite the fact that the proposed confidence sets should be conservative, for large values of p and n the coverage level is very near to the nominal coverage (see also Jankowski and Stanberry, 2012).

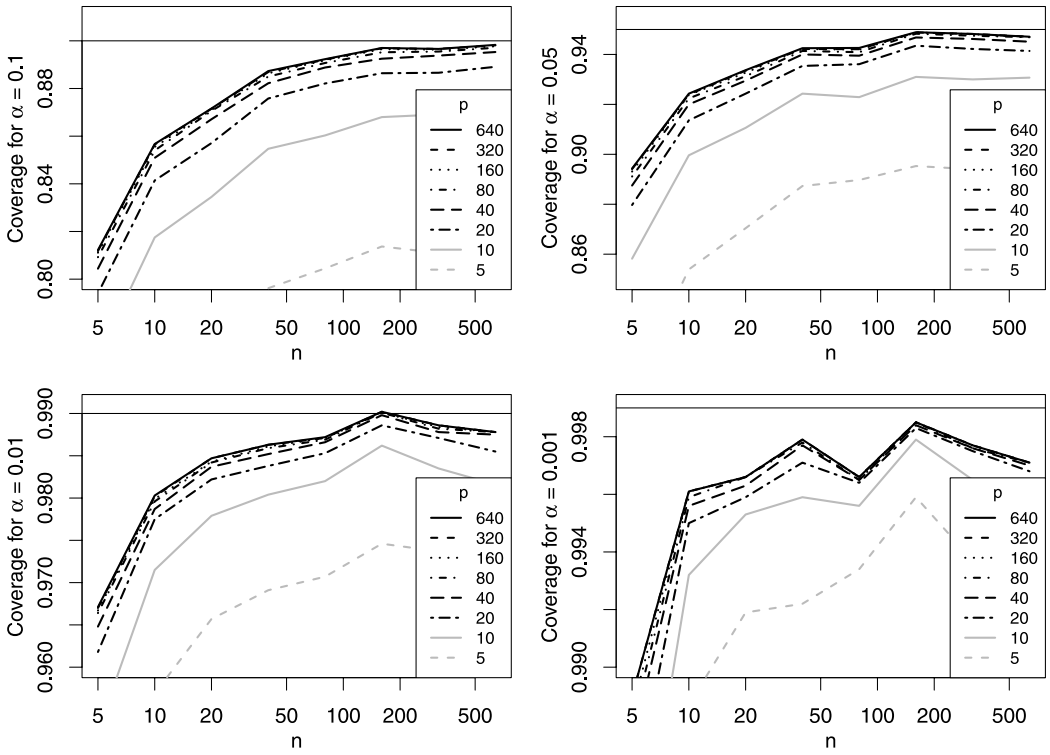

Fig. 4.4. Coverage of one-sided external confidence sets.

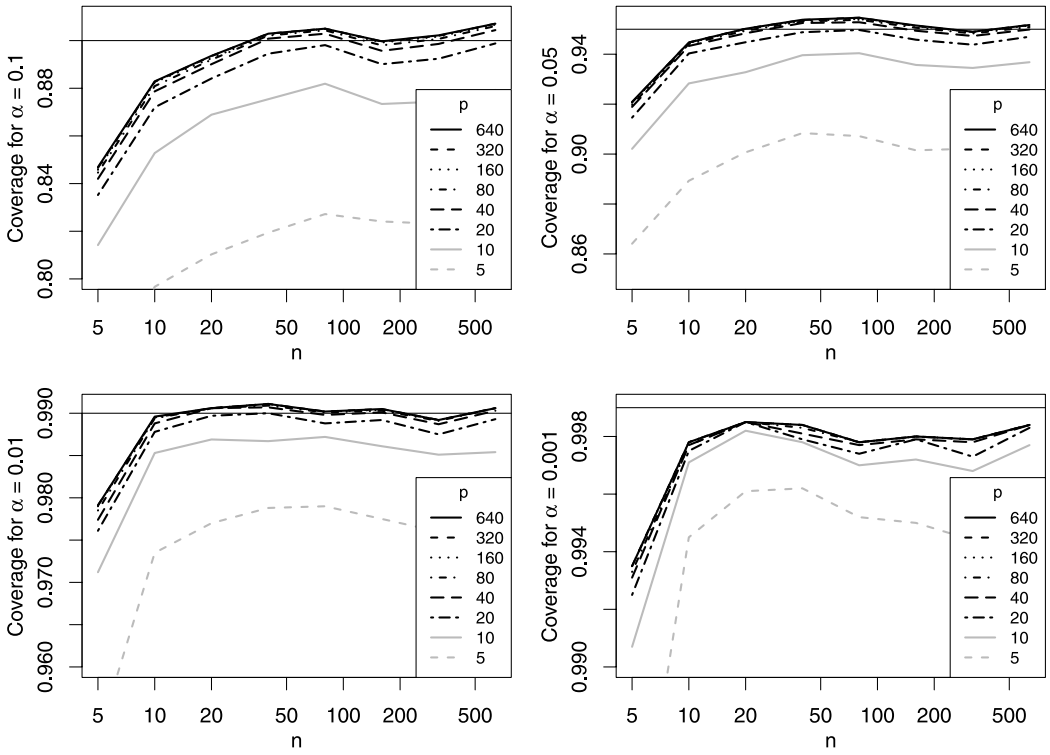

Fig. 4.5. Coverage of one-sided internal confidence sets.

5. Conclusions

In this paper, we propose a general definition of confidence sets for the mean of a random closed set. Even though our main interest is in the Aumann mean, the definition is expected to be of broader interest. It encompasses sets containing the mean, contained in it and containing its boundary with prescribed probability.

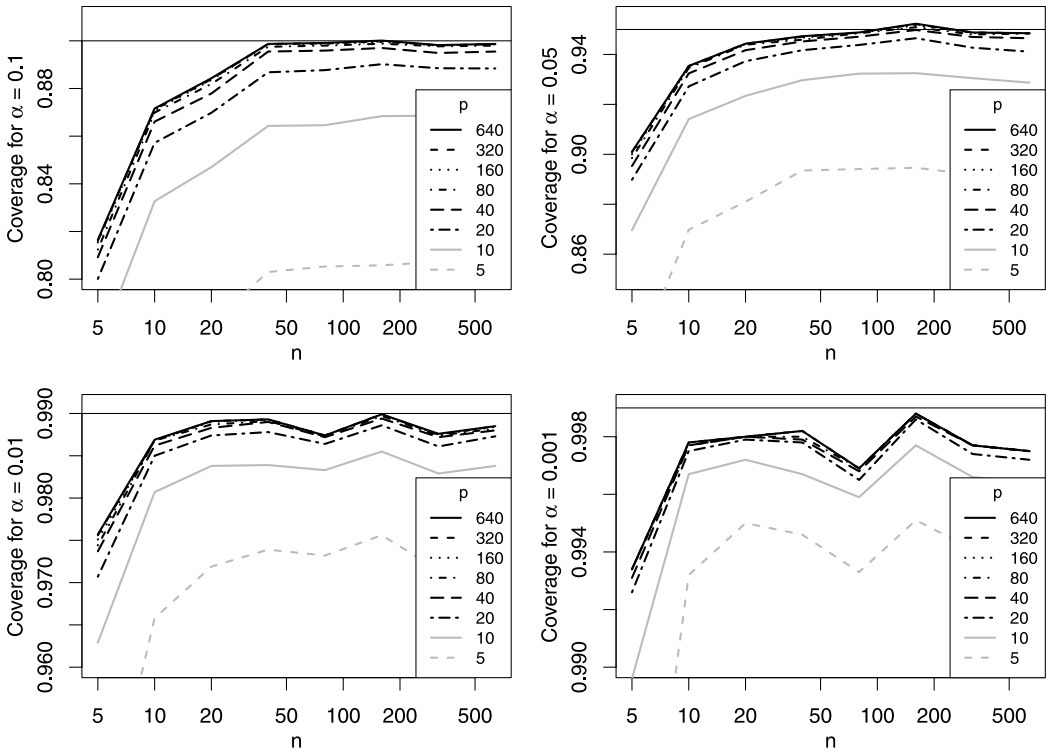

Fig. 4.6. Coverage of two-sided symmetric confidence sets.

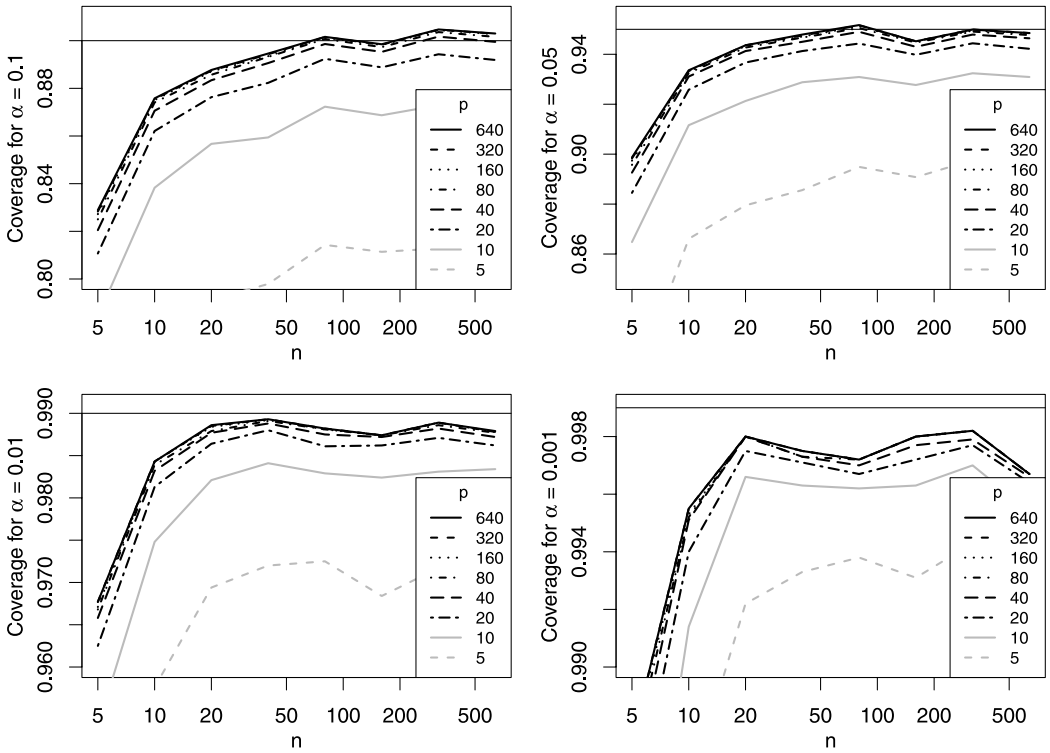

Fig. 4.7. Coverage of two-sided equal-tailed confidence sets.

Then we pass to consider the Aumann mean. Despite having the drawback of always being convex, this definition of the mean of a random closed set is still useful in many cases of interest. We express the confidence sets as parallel bodies of the Minkowski mean with respect to a structuring element. This allows us to provide formulas for the confidence sets in terms of support functions of the original sets. Moving from sets to support functions introduces some conservativeness in confidence sets built in this way. Then we discuss bootstrap implementation of the algorithm. The effectiveness of the

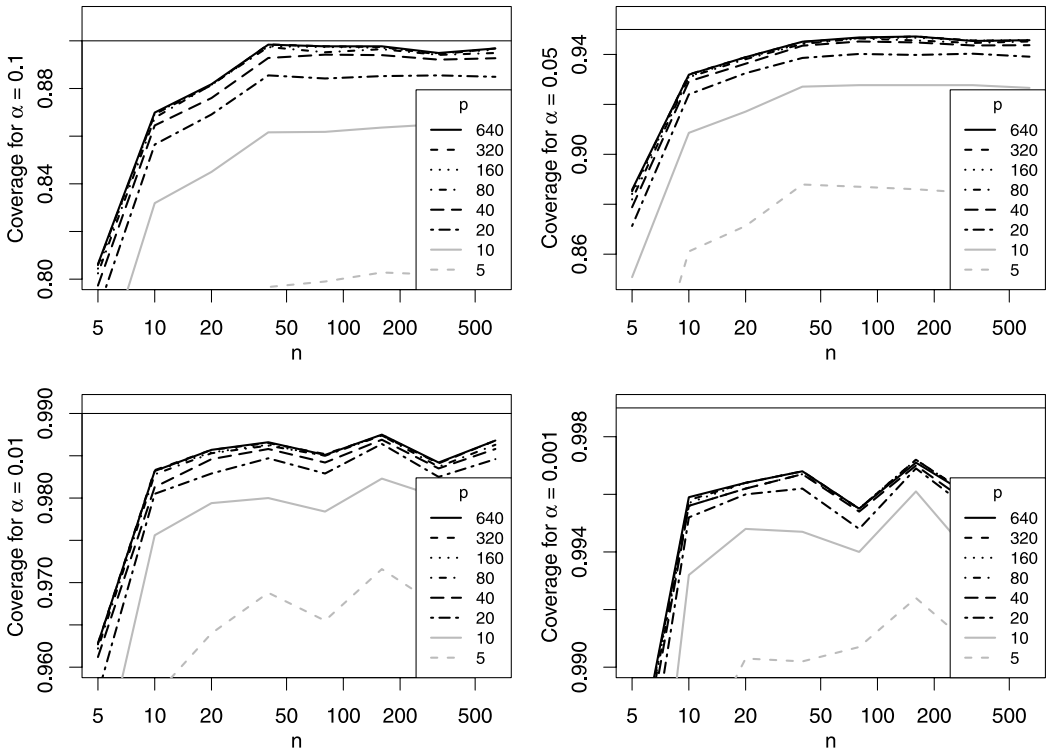

Fig. 4.8. Coverage of two-sided narrowest-width confidence sets.

proposed sets is shown in an illustrative example in which closed-form solutions are available, in a real data situation and in Monte Carlo simulations.

Several points still deserve attention. First of all, moving from sets to support functions is not free from issues, since the support function of the Minkowski subtraction of two sets is not always equal to the subtraction of the support functions of the two sets. This seems to have a limited impact on the coverage level in our Monte Carlo experiments. On the one hand, this is a general consequence of the convexity of the Aumann mean, as the distance between the previous two functions is smaller for rounder sets. On the other hand, this is probably due to our choice of the simulated sets. It would be interesting to have a procedure for building confidence sets that does not suffer from this problem and to better understand when the proposed confidence sets are (asymptotically) exact. Second, theoretical analyses of the performance of the method are still lacking. This seems to be a general drawback of bootstrap for functional data, of which the present case is an instance.

Acknowledgments

We thank Christian Hess, Hanna Jankowski and Ilya Molchanov for useful discussions on the topic of this paper, Dietrich Stoyan for giving us permission to use the sand grains data and Ian Dryden for providing the data in a processable form.

Appendix A. Derivation of λ'_n and λ''_n for Section 4.1

First consider the case in which $B = H$. In this case, the external confidence set is defined by:

$$K''_n = \bar{X}_n \oplus \lambda''_n H = \bar{E}_n H \oplus \lambda''_n H = \left(\bar{E}_n + \lambda''_n\right) \cdot H.$$

Therefore, the equation $\mathbb{P}\left\{\mathbb{E}X \subseteq K''_n\right\} = 1 - \alpha$ can be written as

$$\mathbb{P}\left\{H \subseteq \left(\bar{E}_n + \lambda''_n\right) \cdot H\right\} = 1 - \alpha$$

or $\mathbb{P}\left\{1 - \lambda''_n > \bar{E}_n\right\} = \alpha$. Using the fact that $n \cdot \bar{E}_n \sim \Gamma(n, 1)$, $1 - \lambda''_n = \frac{\gamma_{n,\alpha}}{n}$ and $K''_n = \bar{X}_n \oplus \left(1 - \frac{\gamma_{n,\alpha}}{n}\right) H$. As concerns the internal confidence set, the reasoning is symmetric: from $K'_n = \bar{X}_n \ominus \lambda'_n H = \left(\bar{E}_n - \lambda'_n\right) \cdot H$ (this can be justified from the fact that $\lambda'_n H$ is a summand of $\bar{X}_n$), we get $\mathbb{P}\left\{\left(\bar{E}_n - \lambda'_n\right) \cdot H \subseteq H\right\} = \mathbb{P}\left\{\bar{E}_n - \lambda'_n \leq 1\right\} = 1 - \alpha$, $\lambda'_n = \frac{\gamma_{n,1-\alpha}}{n} - 1$ and $K'_n = \bar{X}_n \ominus \left(\frac{\gamma_{n,1-\alpha}}{n} - 1\right) H$. Remark that the choice $B = H$ yields the smallest possible confidence sets for this definition of X_i.

As concerns the case $B = \mathsf{B}$, we must choose a value of λ'_n and λ''_n such that the sets K'_n and K''_n contain the respective sets with $B = H$. A geometric reasoning should convince the reader that this implies that λ'_n and λ''_n should be such that $\lambda'_n \mathsf{B}$ is contained in $\left(\frac{\gamma_{n,1-\alpha}}{n} - 1\right) H$ and $\lambda''_n \mathsf{B}$ contains $\left(1 - \frac{\gamma_{n,\alpha}}{n}\right) H$: therefore, we have $\lambda'_n = \left(\frac{\gamma_{n,1-\alpha}}{n} - 1\right)$ and $\lambda''_n = \sqrt{2}\left(1 - \frac{\gamma_{n,\alpha}}{n}\right)$.

Now, we turn to the confidence sets based on the support function. We recall the formulas:

$$\alpha = \mathbb{P}\left\{\inf_y\left(\frac{h_{\bar{X}_n} - h_{\mathbb{E}X}}{h_B}\right) < -\lambda_n''\right\},$$
$$1-\alpha = \mathbb{P}\left\{\sup_y\left(\frac{h_{\bar{X}_n} - h_{\mathbb{E}X}}{h_B}\right) \leq \lambda_n'\right\},$$

respectively for the quantiles of the external and internal set. If we take $B = H$, we use the previous formulas for $h_{\bar{X}_n}$, $h_{\mathbb{E}X}$ and $h_H = \sqrt{2}\cdot\max_{\theta\in\left\{\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}\right\}}\cos(x-\theta)$, and we get $\alpha = \mathbb{P}\left\{\bar{E}_n - 1 < -\lambda_n''\right\}$ and $\lambda_n'' = 1 - \frac{\gamma_{n,\alpha}}{n}$, and $1-\alpha = \mathbb{P}\left\{\bar{E}_n - 1 \leq \lambda_n'\right\}$ and $\lambda_n' = \frac{\gamma_{n,1-\alpha}}{n} - 1$. On the other hand, if we take $B = \mathsf{B}$, we have $h_{\mathsf{B}} = 1$ and:

$$\alpha = \mathbb{P}\left\{\inf_y\left(\sqrt{2}\left(\bar{E}_n - 1\right)\cdot\max_{\theta\in\left\{\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}\right\}}\cos(y-\theta)\right) < -\lambda_n''\right\}.$$

Now:

$$\inf_y\left(\sqrt{2}\left(\bar{E}_n - 1\right)\cdot\max_{\theta\in\left\{\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}\right\}}\cos(y-\theta)\right) = \min\left\{\sqrt{2}\left(\bar{E}_n - 1\right), \left(\bar{E}_n - 1\right)\right\}.$$

Provided α is small enough:

$$\begin{aligned}\alpha &= \mathbb{P}\left\{\min\left\{\sqrt{2}\left(\bar{E}_n - 1\right), \left(\bar{E}_n - 1\right)\right\} < -\lambda_n''\right\}\\ &= \mathbb{P}\left\{\sqrt{2}\left(\bar{E}_n - 1\right) < -\lambda_n''\right\}\\ &= \mathbb{P}\left\{\Gamma(n,1) < n\cdot\left(1 - \frac{\lambda_n''}{\sqrt{2}}\right)\right\},\end{aligned}$$

where we have used the fact that $n\cdot\bar{E}_n \sim \Gamma(n,1)$. At last, we get $n\left(1 - \frac{\lambda_n''}{\sqrt{2}}\right) = \gamma_{n,\alpha}$, from which $\lambda_n'' = \sqrt{2}\left(1 - \frac{\gamma_{n,\alpha}}{n}\right)$. As concerns the internal set, we have:

$$\sup_y\left(\sqrt{2}\left(\bar{E}_n - 1\right)\cdot\max_{\theta\in\left\{\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}\right\}}\cos(y-\theta)\right) = \max\left\{\sqrt{2}\left(\bar{E}_n - 1\right), \left(\bar{E}_n - 1\right)\right\}.$$

Provided α is small enough, $n\left(\frac{\lambda_n'}{\sqrt{2}} + 1\right) = \gamma_{n,1-\alpha}$ and $\lambda_n' = \sqrt{2}\left(\frac{\gamma_{n,1-\alpha}}{n} - 1\right)$.

Appendix B. Derivation of the Aumann mean for Section 4.3

Let X be the convex hull of 5 points $\{x_1,\ldots,x_5\}$ in $\mathbb{R}^2$ drawn from a bivariate standard normal distribution. In this appendix, we show that $\mathbb{E}X$ is given by $1.1629644736\cdot\mathsf{B}$, the ball with radius 1.1629644736. Indeed, pick any direction $y\in\mathbb{S}^1$. From Proposition 5 (b) in Ghosh and Kumar (1998), $h_X(y) = \max_{1\leq i\leq 5} h_{\{x_i\}}(y) = \max_{1\leq i\leq 5}\langle x_i, y\rangle$. Since the 5 points are drawn from a bivariate standard normal distribution, $h_X(y)$ is the maximum of 5 iid standard normal random variables and $\mathbb{E}h_X(y)$ is the expected value of the largest order statistic from such a sample, i.e. 1.1629644736 (see Teichroew, 1956). This shows that $h_{\mathbb{E}X}$ is constant. Since $\mathbb{E}h_X = h_{\mathbb{E}X}$, $h_{\mathbb{E}X}$ is constant and $\mathbb{E}X$ is convex, $\mathbb{E}X$ is a ball of radius 1.1629644736.

Appendix C. Supplementary data

Supplementary material related to this article can be found online at http://dx.doi.org/10.1016/j.csda.2012.10.015.

References

Artstein, Z., Hansen, J., 1985. Convexification in limit laws of random sets in Banach spaces. Ann. Probab. 13, 307–309.
Artstein, Z., Vitale, R., 1975. A strong law of large numbers for random compact sets. Ann. Probab. 3, 879–882.
Beresteanu, A., Molinari, F., 2008. Asymptotic properties for a class of partially identified models. Econometrica 76, 763–814.
Choirat, C., Hess, C., Seri, R., 2003. A functional version of the Birkhoff ergodic theorem for a normal integrand: a variational approach. Ann. Probab. 31, 63–92.
Choirat, C., Seri, R., 2013. Computational aspects of Cui–Freeden statistics for equidistribution on the sphere. Math. Comp. (electronic) (forthcoming).
Cressie, N., 1979. A central limit theorem for random sets. Z. Wahrscheinlichkeitstheor. Verwandte Geb. 49, 37–47.
Cuevas, A., Febrero, M., Fraiman, R., 2006. On the use of the bootstrap for estimating functions with functional data. Comput. Statist. Data Anal. 51, 1063–1074.

Cui, J., Freeden, W., 1997. Equidistribution on the sphere. SIAM J. Sci. Comput. 18, 595–609.
Dryden, I., 2012. shapes package. R Foundation for Statistical Computing. Vienna, Austria. Contributed package.
Ghosh, P., Kumar, K., 1998. Support function representation of convex bodies, its application in geometric computing, and some related representations. Comput. Vis. Image Underst. 72, 379–403.
Gil, M.Á, Montenegro, M., González-Rodríguez, G., Colubi, A., Casals, M.R., 2006. Bootstrap approach to the multi-sample test of means with imprecise data. Comput. Statist. Data Anal. 51, 148–162.
Giné, E., Hahn, M., Zinn, J., 1983. Limit theorems for random sets: an application of probability in Banach space results. In: Probability in Banach Spaces, IV (Oberwolfach, 1982). In: Lecture Notes in Math., vol. 990. Springer, Berlin, pp. 112–135.
Giné, E., Zinn, J., 1990. Bootstrapping General Empirical Measures. Ann. Probab. 18, 851–869.
González-Rodríguez, G., Colubi, A., Gil, M.Á, 2012. Fuzzy data treated as functional data: a one-way ANOVA test approach. Comput. Statist. Data Anal. 56, 943–955.
González-Rodríguez, G., Trutschnig, W., Colubi, A., 2009. Confidence regions for the mean of a fuzzy random variable, in: Carvalho, J.P., Dubois, D., Kaymak, U., da Costa Sousa, J.M. (Eds.), IFSA/EUSFLAT Conf., pp. 1433–1438.
Hall, P., Pittelkow, Y., 1990. Simultaneous bootstrap confidence bands in regression. J. Stat. Comput. Simul. 37, 99–113.
Jankowski, H., Stanberry, L., 2010. Expectations of random sets and their boundaries using oriented distance functions. J. Math. Imaging Vision 36, 291–303.
Jankowski, H., Stanberry, L., 2012. Confidence regions for means of random sets using oriented distance functions. Scand. J. Stat. 39, 340–357.
Kendall, D., 1974. Foundations of a theory of random sets. In: Stochastic Geometry (A Tribute to the Memory of Rollo Davidson). Wiley, London, pp. 322–376.
Kent, J., Dryden, I., Anderson, C., 2000. Using circulant symmetry to model featureless objects. Biometrika 87, 527–544.
Ljašenko, N., 1979. Limit theorems for sums of independent compact random subsets of a Euclidean space. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 85, 113–128, 240, 246.
Matheron, G., 1972. Ensembles fermés aléatoires, ensembles semi-markoviens et polyèdres Poissoniens. Adv. Appl. Probab. 4, 508–541.
Matheron, G., 1975. Random Sets and Integral Geometry. John Wiley & Sons, New York-London-Sydney.
McMurry, T., Politis, D., 2011. Resampling methods for functional data. In: F., Ferraty, Y., Romain (Eds.), The Oxford Handbook of Functional Data Analysis. Oxford University Press, pp. 189–209.
Molchanov, I., 1997. Statistics of the Boolean Model for Practitioners and Mathematicians. Wiley, Chichester.
Molchanov, I., 2005. Theory of Random Sets. In: Probability and its Applications (New York), Springer-Verlag London Ltd., London.
Molchanov, I., 2010. Random sets in finance and econometrics. In: New Perspectives in Stochastic Geometry. Oxford Univ. Press, Oxford, pp. 555–577.
Montenegro, M., Colubi, A., Casals, M.R., Gil, M.Á., 2004. Asymptotic and bootstrap techniques for testing the expected value of a fuzzy random variable. Metrika 59, 31–49.
Puri, M., Ralescu, D., 1985. Limit theorems for random compact sets in Banach space. Math. Proc. Cambridge Philos. Soc. 97, 151–158.
R Development Core Team, 2011. R: A Language and Environment for Statistical Computing. R Foundation for Statistical Computing, Vienna, Austria, ISBN: 3-900051-07-0.
Ramos-Guajardo, A.B., Lubiano, M.A., 2012. K-sample tests for equality of variances of random fuzzy sets. Comput. Statist. Data Anal. 56, 956–966.
Schneider, R., 1993. Convex Bodies: The Brunn–Minkowski Theory. In: Encyclopedia of Mathematics and its Applications, vol. 44. Cambridge University Press, Cambridge.
Seri, R., Choirat, C., 2004. Confidence sets for the Aumann mean of a random closed set. In: Computational Science and Its Applications — ICCSA 2004. Part III. In: Lecture Notes in Comput. Sci., vol. 3045. Springer, Berlin, pp. 298–307.
Stoyan, D., 1997. Geometrical means, medians and variances for samples of particles. Part. Part. Syst. Charact. 14, 30–34.
Teichroew, D., 1956. Tables of expected values of order statistics and products of order statistics for samples of size twenty and less from the normal distribution. Ann. Math. Statist. 27, 410–426.
Weil, W., 1982. An application of the central limit theorem for Banach-space-valued random variables to the theory of random sets. Z. Wahrscheinlichkeitstheor. Verwandte Geb. 60, 203–208.