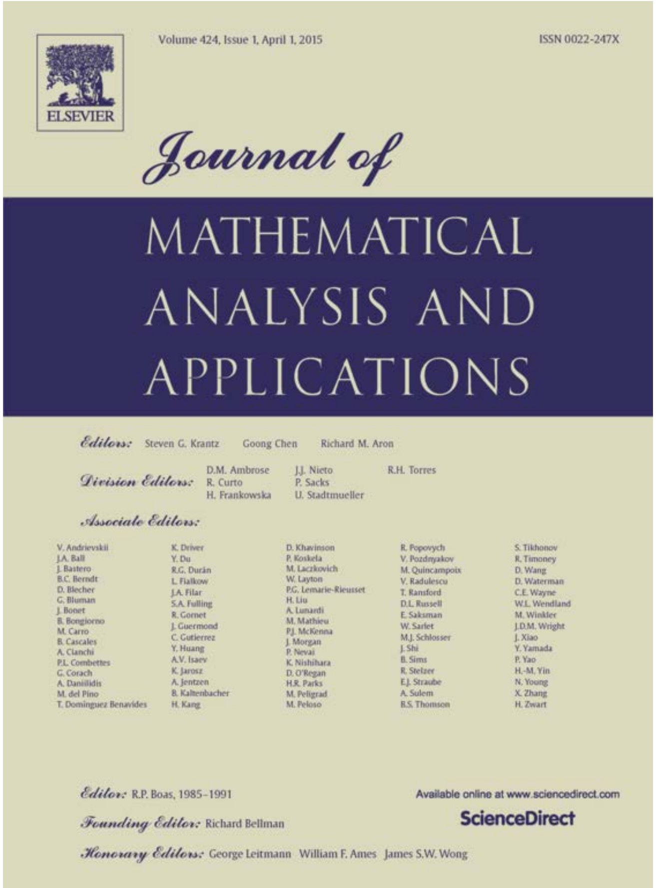
Volume 424, Issue 1, April 1, 2015
ISSN 0022-247X
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Journal of
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J. Math. Anal. Appl. 424 (2015) 826–834

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Journal of Mathematical Analysis and Applications

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Note

A non-recursive formula for the higher derivatives of the Hurwitz zeta function

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ARTICLE INFO

Article history:
Received 19 December 2013
Available online 19 August 2014
Submitted by B.C. Berndt

Keywords:
Hurwitz zeta function
Asymptotic formula
Stieltjes coefficients

ABSTRACT

In this paper, we provide an asymptotic formula for the higher derivatives of the Hurwitz zeta function with respect to its first argument that does not need recurrences. As a by-product, we correct some formulas that have appeared in the literature.

1. Introduction

In this paper, we provide an asymptotic formula (see Eq. (4) below) for the higher derivatives of the Hurwitz zeta function with respect to its first argument that does not need recurrences. In passing by, we correct some minor slips in formulas that have been advanced in [17] and we provide an asymptotic formula for the Stieltjes coefficients (see [12, Proposition 3], for a more general asymptotic expression).

We recall that the *Hurwitz zeta* or *generalized Riemann zeta function* is defined as (see, e.g., [18, p. 3])

$$\zeta(z,a) := \sum_{n=0}^{\infty} (n+a)^{-z}, \quad \Re(z) > 1, \ a \neq 0, -1, -2, \dots$$

The function has a simple pole in $z = 1$ with residue 1 and can be analytically continued to the rest of the complex plane. In the following we will indicate its i-th derivative with respect to its first argument as $\zeta^{(i)}$ (it is intended that $\zeta \equiv \zeta^{(0)}$, $\zeta' \equiv \zeta^{(1)}$ and $\zeta'' \equiv \zeta^{(2)}$). Moreover, we set $c_{m,j}(z,a) := -\binom{m}{j}\left(\frac{j}{z} + \ln a\right) \cdot \ln^{j-1} a$ (often abbreviated as $c_{m,j}$) for $m \geq 1$ and $1 \leq j \leq m$.

Several references have considered formulas for ζ' when $z = 0$ ([30, pp. 22–25], [21, p. 26] and [22, pp. 1072–1074]; see also [6, p. 121], [26, p. 169] and [25, p. 651]), when $z = -1$ ([14, Eq. (3.1)], [20, Eq. (2.39)], [15,34]), when $-z \in \mathbb{N}$ [16,1,7,3], for some special values of a [19], when a is rational [31], when $\Re(z) > \frac{1}{2}$ [9]. The first derivative of ζ has also been linked to some integrals involving cyclotomic polynomials

http://dx.doi.org/10.1016/j.jmaa.2014.08.012

and iterated logarithms in [1], polygamma functions of negative order in [2], the multiple gamma function in [8,3,4], and a log-gamma integral in [19] and in [7].

The second derivative with $z = 0$ has been studied in [13] (see also [26, p. 169] and [25, p. 651]). In [17] (see also [18, Chapter 2]) recursive formulas for the higher derivatives and direct formulas for the case of small m and small $-z \in \mathbb{N}$ are considered. Higher derivatives with $z = 0$ have been considered in [3] and [32]. In [23] high-precision computation of the Hurwitz zeta function and its derivatives is discussed; the formulas have been implemented in the Python library mpmath (see [24]) that we will use below. Higher order derivatives of the Hurwitz zeta function appear in formulas for generalized Stieltjes constants as shown in [10, Proposition 8]. In [27], the authors derive formulas for the mean square of higher derivatives of Hurwitz zeta functions.

We first recall some formulas that will be used in the following. We know that (see, e.g., [17, p. 3223]):

$$\zeta(z+1,a) = \frac{1}{z}a^{-z} + \frac{1}{2}a^{-z-1} + \frac{1}{z}\Sigma_0(z,a)$$
$$\Sigma_0(z,a) = \sum_{k=2}^{\infty} B_k a^{-z-k}\frac{(z)_k}{k!}. \tag{1}$$

From [16] and [17, p. 3223]

$$\zeta'(z+1,a) = -\left(\frac{1}{z} + \ln a\right)\zeta(z+1,a) + \frac{1}{2z}a^{-z-1} + z^{-1}\Sigma_1(z,a) \tag{2}$$

where

$$\Sigma_1(z,a) := \sum_{k=2}^{\infty} B_k a^{-z-k} \sum_{j=0}^{k-1} \frac{(z)_j}{j!(k-j)}.$$

From [17, Eq. (13)], for $m \geq 2$,

$$\zeta^{(m)}(z+1,a) = \sum_{j=1}^{m} c_{m,j}(z,a)\cdot\zeta^{(m-j)}(z+1,a) + z^{-1}\Sigma_m(z,a) \tag{3}$$

where

$$\Sigma_m(z,a) := \sum_{j_0=2}^{\infty} B_{j_0} a^{-z-j_0}\left\{\sum_{j_1=0}^{j_0-1}\frac{1}{j_0-j_1}\sum_{j_2=0}^{j_1-1}\frac{1}{j_1-j_2}\cdots\sum_{j_m=0}^{j_{m-1}-1}\frac{(z)_{j_m}}{j_m!(j_{m-1}-j_m)}\right\},$$

the B_k's are the Bernoulli numbers and $(n)_k := n(n+1)\cdots(n+k-1) = \frac{\Gamma(n+k)}{\Gamma(n)}$ is Pochhammer's symbol. Remark that the formula for ζ' cannot be obtained from (3) simply setting $m = 1$ as it contains an additional term in a^{-z-1}.

In what follows, empty products are taken as equal to 1 and empty sums to 0. Our formula, for $m \geq 2$, large $|a|$ and $|\arg a| < \pi$, is:

$$\begin{aligned}\zeta^{(m)}&(z+1,a)\\ &= \frac{(-1)^m\Gamma(m+1,z\ln a)}{z^{m+1}} + \frac{(-1)^m}{2}\cdot\ln^m a\cdot a^{-z-1} + \frac{1}{z}\Sigma_m(z,a)\\ &\quad + \frac{1}{z}\sum_{i=0}^{m-1}\Sigma_i(z,a)\cdot\left\{c_{m,m-i} + \sum_{\ell=1}^{m-i-1}\sum_{1\leq k_0<\cdots<k_{\ell-1}<m-i}\left[\prod_{j=0}^{\ell-1}c_{m-k_{j-1},k_j-k_{j-1}}\right]c_{m-k_{\ell-1},m-i-k_{\ell-1}}\right\}\end{aligned} \tag{4}$$

where it is intended that $k_{-1} = 0$. In Section 2 the result will be applied to obtain formulas for $m = 2$ and $z = -1, -2, -3, -4$ and to derive an asymptotic formula for the Stieltjes coefficients. In Section 3 we provide a numerical illustration of the accuracy of the proposed formulas, while Section 4 contains the proof.

2. Applications

In this section we provide two very simple applications of this result.

First of all, we correct some minor slips contained in Eqs. (19)–(22) in [17]. For $z = -1, -2, -3, -4$, the correct formulas are:

$$\begin{aligned}
\zeta''(0,a) &= -\left(\ln^2 a - 2\cdot\ln a + 2\right)a + \frac{\ln^2 a}{2} - \frac{\ln a}{6}a^{-1} \\
&\quad + \left(\frac{\ln a}{180} - \frac{1}{120}\right)a^{-3} - \left(\frac{\ln a}{630} - \frac{5}{1512}\right)a^{-5} + \cdots, \\
\zeta''(-1,a) &= -\left(\frac{\ln^2 a}{2} - \frac{\ln a}{2} + \frac{1}{4}\right)a^2 + \frac{\ln^2 a}{2}a - \left(\frac{\ln^2 a}{12} + \frac{\ln a}{6}\right) \\
&\quad - \frac{\ln a}{360}a^{-2} + \left(\frac{\ln a}{2520} - \frac{1}{3024}\right)a^{-4} + \cdots, \\
\zeta''(-2,a) &= -\left(\frac{\ln^2 a}{3} - \frac{2\ln a}{9} + \frac{2}{27}\right)a^3 + \frac{\ln^2 a}{2}a^2 - \left(\frac{\ln^2 a}{6} + \frac{\ln a}{6}\right)a \\
&\quad + \left(\frac{\ln a}{180} + \frac{1}{120}\right)a^{-1} - \frac{\ln a}{3780}a^{-3} + \left(\frac{\ln a}{12\,600} - \frac{1}{21\,600}\right)a^{-5} + \cdots, \\
\zeta''(-3,a) &= -\left(\frac{\ln^2 a}{4} - \frac{\ln a}{8} + \frac{1}{32}\right)a^4 + \frac{\ln^2 a}{2}a^3 - \left(\frac{\ln^2 a}{4} + \frac{\ln a}{6}\right)a^2 \\
&\quad + \left(\frac{\ln^2 a}{120} + \frac{11\ln a}{360} + \frac{1}{60}\right) + \left(\frac{\ln a}{2520} + \frac{1}{3024}\right)a^{-2} - \frac{\ln a}{16\,800}a^{-4} + \cdots
\end{aligned}$$

These formulas agree with the ones obtained through the heuristic approach of differentiating a truncated version of the infinite sum (1).

Then we show how the result can be applied to provide an asymptotic formula in $|a|$ for the Stieltjes coefficients. The Laurent series around $z = 1$ can be written as:

$$\zeta(z,a) = \frac{1}{z-1} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n(a)}{n!} \cdot (z-1)^n$$

where the $\gamma_n(a)$'s are called the Stieltjes (or generalized Euler) coefficients. The most interesting asymptotics is clearly the one for $n \to \infty$, extensively covered in [29]; see also [28] and [5]. An asymptotic result for $a \to 0$ is contained in Proposition 5 in [11]. Here we only sketch a result providing an asymptotic series for $|a| \to \infty$ with $|\arg a| < \pi$, that is contained in Proposition 3 in [12]. From Proposition 8 in [10], using (4), we get:

$$\begin{aligned}
\gamma_n(a) &= -\frac{\ln^{n+1}(a+1)}{n+1} + \frac{\ln^n a}{a} - (-1)^n \sum_{k=1}^{\infty} \frac{1}{k+1} \cdot \left[(-1)^k \cdot \zeta^{(n)}(k+1, a+1)\right. \\
&\quad \left. - \frac{n!}{k!} \cdot \sum_{j=0}^{n-1} \frac{(-1)^j \cdot s(k+1, j+2)}{(n-j-1)!} \cdot \zeta^{(n-j-1)}(k+1, a+1)\right] \\
&\sim -\frac{\ln^{n+1}(a+1)}{n+1} + \frac{\ln^n a}{2a}
\end{aligned}$$

where the notation $s(n,k)$ denotes the Stirling numbers of the first kind (see, e.g., [33, p. 624]).

3. Numerical example

Now we provide a numerical demonstration of the usefulness of the proposed asymptotic expression for large $|a|$. In Tables 1, 2 and 3 we compute to 10-digit accuracy the ratio of two truncated expansions taken from (4) to $\zeta^{(m)}(z+1,a)$ for $a \in \{2,4,8,16,32,64,128,256,512,1024\}$, $m \in \{2,10,50\}$ and $z \in \{-100,-10,-1,1,10,100\}$. The computations have been performed in Python using the mpmath library developed by Fredrik Johansson (see [24]), that allows for arbitrary-precision floating-point computations of, among others, the Hurwitz zeta function and its derivatives. The results show that $\frac{(-1)^m \Gamma(m+1,z\ln a)}{z^{m+1}} + \frac{(-1)^m}{2} \cdot \ln^m a \cdot a^{-z-1}$ generally improves over $\frac{(-1)^m \Gamma(m+1,z\ln a)}{z^{m+1}}$. The asymptotic expansion is better for moderate values of $|z|$ than for extreme ones, and for positive values of z than for negative ones, even if this becomes evident only for very large m. The dependence on a is less simple to characterize: despite the expansion is asymptotic and, as such, it should be better for large a, the accuracy is far from monotonically increasing.

4. Proof

In the following, we will write $\zeta^{(m)}(z+1,a) = \widetilde{\zeta}_0^{(m)}(z+1,a) + \widetilde{\zeta}_1^{(m)}(z+1,a) + z^{-1}\widetilde{\Sigma}_m(z,a)$ where, for $m \geq 0$, $\widetilde{\Sigma}_m(z,a)$ contains only the terms multiplying a^{-z-k} with $k \geq 2$, while $\widetilde{\zeta}_0^{(m)}(z+1,a)$ and $\widetilde{\zeta}_1^{(m)}(z+1,a)$ contain respectively those multiplying a^{-z} and a^{-z-1}. An inspection of formula (3) shows that this rewriting is indeed possible. Therefore, in order to have equality for any z and a, we can split (3) in three parts:

$$\widetilde{\zeta}_0^{(m)}(z+1,a) = \sum_{j=1}^{m} c_{m,j}(z,a) \cdot \widetilde{\zeta}_0^{(m-j)}(z+1,a) \tag{5}$$

$$\widetilde{\zeta}_1^{(m)}(z+1,a) = \sum_{j=1}^{m} c_{m,j}(z,a) \cdot \widetilde{\zeta}_1^{(m-j)}(z+1,a) \tag{6}$$

$$z^{-1}\widetilde{\Sigma}_m(z,a) = \sum_{j=1}^{m} c_{m,j}(z,a) \cdot z^{-1}\widetilde{\Sigma}_{m-j}(z,a) + z^{-1}\Sigma_m(z,a). \tag{7}$$

As concerns the terms (5) and (6), we write them singling out the 0-th and the 1-st derivatives:

$$\begin{aligned}\widetilde{\zeta}_i^{(m)}(z+1,a) = &-\sum_{j=1}^{m-2} \binom{m}{j}\left(\frac{j}{z} + \ln a\right) \cdot \ln^{j-1} a \cdot \widetilde{\zeta}_i^{(m-j)}(z+1,a) \\ &- m\left(\frac{m-1}{z} + \ln a\right) \cdot \ln^{m-2} a \cdot \widetilde{\zeta}_i'(z+1,a) \\ &- \left(\frac{m}{z} + \ln a\right) \cdot \ln^{m-1} a \cdot \widetilde{\zeta}_i(z+1,a), \quad i = 0,1.\end{aligned} \tag{8}$$

As concerns (5), $\widetilde{\zeta}_0^{(m)}(z+1,a)$ contains only the power a^{-z} multiplied by $\ln^j a$ for $0 \leq j \leq m$, so that we suppose without loss of generality that it has the form:

$$\widetilde{\zeta}_0^{(m)}(z+1,a) = (-1)^m \cdot \sum_{h=0}^{m} \frac{a_{h,m}(z)}{z^{h+1}} \cdot a^{-z} \cdot \ln^{m-h} a, \quad m > 1,$$

for an adequate choice of $a_{h,m}(z)$ with $0 \leq h \leq m$. We inject this into (8) and we equate the powers of $\ln a$. This yields, for the k-th power of $\ln a$,

Table 1
Ratio of first order ($\frac{(-1)^m \Gamma(m+1, z \ln a)}{z^{m+1}}$) and second order ($\frac{(-1)^m \Gamma(m+1, z \ln a)}{z^{m+1}} + \frac{(-1)^m}{2} \cdot \ln^m a \cdot a^{-z-1}$) terms to $\zeta^{(m)}(z+1, a)$ for $m = 2$, $a \in \{2, 4, 8, 16, 32, 64, 128, 256, 512, 1024\}$ and $z \in \{-100, -10, -1, 1, 10, 100\}$.

First order term

a \ z	−100	−10	−1	1	10	100
2	−4.03035319492646e−50	−2232.14828625428	1.0906921844083	0.971896092960699	0.257966453337044	0.0205854034922796
4	−2.07332150442978e−19	7.27183126798027	1.24450296900883	0.964576849403802	0.40483050219671	0.0405812407454039
8	49 861.2988697718	2.21129075370777	1.1393637411514	0.974338306695301	0.594286362694693	0.080772522778146
16	99.0473009480427	1.43264451948172	1.06107657980635	0.98427065966303	0.756643017420649	0.16078914451995
32	7.22505861909308	1.18663186677276	1.02714636943983	0.991052958321174	0.865440162581429	0.307142770068375
64	2.4341934047327	1.0868961330867	1.01244590200146	0.995111234486079	0.928939217431552	0.506863924936852
128	1.51965186086875	1.0418912911501	1.00583710403715	0.99739230389584	0.96338908240265	0.694715801903155
256	1.22454323858552	1.0205383240575	1.00277992727319	0.998630490863419	0.981383166781768	0.828276351293884
512	1.10473349285675	1.01015522332537	1.00133795361848	0.999288332652858	0.990598787579293	0.908651601569673
1024	1.05061658725134	1.00504344315204	1.00064876195268	0.999632949685508	0.995270250163831	0.952847952818615

Second order term

a \ z	−100	−10	−1	1	10	100
2	9.96776872515917e−49	5177.83296859918	0.970959468792559	1.00208622188486	0.742805146645382	0.520585403492279
4	2.42197904396747e−18	−3.22279661217825	0.98435979888006	0.999190392111077	0.843081556863934	0.54058124063558
8	−264 783.50927434	0.689948442556273	0.997150832195004	0.999457129887912	0.931702162062295	0.580768717128546
16	−212.716302347946	0.951486743893736	0.999537621630211	0.999793328713881	0.976651318082907	0.659643766938644
32	−4.12943067175208	0.990212780390045	0.999918164017075	0.999934188637623	0.993086457676591	0.78429188069072
64	0.523312508466132	0.997800815966444	0.999984394043228	0.999980736924943	0.998106377403235	0.900951648047943
128	0.923585941250452	0.999480265932024	0.999996865449626	0.999994619890835	0.999502210905976	0.964972870277512
256	0.984510463302829	0.999874070103545	0.999999347082069	0.999998539926149	0.999871939623729	0.989466656594165
512	0.996502932929171	0.999969094905442	0.999999860354432	0.999999611158471	0.999967431456758	0.997103080500456
1024	0.999168716189077	0.999992363752758	0.999999969530938	0.999999897783649	0.999991768515609	0.999239680417542

Table 2
Ratio of first order ($\frac{(-1)^m \Gamma(m+1, z \ln a)}{z^{m+1}}$) and second order ($\frac{(-1)^m \Gamma(m+1, z \ln a)}{z^{m+1}} + \frac{(-1)^m}{2} \cdot \ln^m a \cdot a^{-z-1}$) terms to $\zeta^{(m)}(z+1, a)$ for $m = 10$, $a \in \{2, 4, 8, 16, 32, 64, 128, 256, 512, 1024\}$ and $z \in \{-100, -10, -1, 1, 10, 100\}$.

First order term

a \ z	−100	−10	−1	1	10	100
2	−2.27559728463351e−55	6.40026218639767	1.00000000099006	0.999999999759553	0.882059654763553	0.0233080501978386
4	−3.15631464883507e−22	31.0954669619979	1.00000254916215	0.999999823165264	0.649963278266152	0.0430840936153883
8	81 666.2812870931	3.09002538350336	1.0001873899748	0.99999693596817	0.719370745297742	0.0840193305132564
16	116.264597427748	1.59364521198324	1.0034879840561	0.999985791442525	0.818989981816743	0.165511585154218
32	7.62424982420115	1.23437572747313	1.0225775696246	0.999966484197275	0.894790281536294	0.313314186069497
64	2.48093169155181	1.10423770409183	1.02519726687069	0.999947786665193	0.942301836408273	0.512619135272541
128	1.53080743460235	1.04884948840905	1.01248758619215	0.999938502920964	0.969394652938018	0.698626020699342
256	1.22823596678992	1.02348507368061	1.00572943489663	0.999940624961421	0.984078589354975	0.830460371736268
512	1.1061662402362	1.01144349492191	1.00264794621749	0.999950437740476	0.991813580554902	0.909752104768885
1024	1.05121953334584	1.00561821295249	1.00123752882471	0.999962878429022	0.995821210336008	0.953375733748238

Second order term

a \ z	−100	−10	−1	1	10	100
2	6.29276481761246e−54	−34.1121329724027	0.999999997462604	1.00000000064142	1.04969708558877	0.523308050197838
4	3.91630111726862e−21	−37.0464545649718	0.999998937049974	1.00000004892245	0.952619472772679	0.54308409325294
8	−453 407.336868769	0.198474052090923	0.999979169815996	1.00000019057086	0.968304591569069	0.584013416885132
16	−260.212444121979	0.911055158639644	0.999863855243712	1.00000024187242	0.987114234089732	0.664149652294443
32	−4.63334317060652	0.984562256459378	0.999655527004422	1.00000015526849	0.995739699586942	0.788783790285799
64	0.495989344012472	0.996810755500282	0.99984122794861	1.0000000550567	0.998737253995231	0.903496843128416
128	0.920486640895884	0.999286238861439	0.999966398316669	1.00000000488423	0.999647910706374	0.965914129489308
256	0.984012340883862	0.999833652687128	0.999993265344664	0.999999992400064	0.999905242379091	0.989739963515607
512	0.996407833311745	0.999960366238282	0.999998617558798	0.99999999359572	0.999975034941971	0.99717324814074
1024	0.999148876376108	0.999990434717137	0.999999709343093	0.999999996633112	0.99999350978819	0.99925666653692

Table 3
Ratio of first order ($\frac{(-1)^m \Gamma(m+1, z \ln a)}{z^{m+1}}$) and second order ($\frac{(-1)^m \Gamma(m+1, z \ln a)}{z^{m+1}} + \frac{(-1)^m}{2} \cdot \ln^m a \cdot a^{-z-1}$) terms to $\zeta^{(m)}(z+1, a)$ for $m = 50$, $a \in \{2, 4, 8, 16, 32, 64, 128, 256, 512, 1024\}$ and $z \in \{-100, -10, -1, 1, 10, 100\}$.

First order term

a \ z	−100	−10	−1	1	10	100
2	−1.20784866953287e−82	1.0000344358472	1.0	1.0	1.0	0.0649945450278254
4	−2.1854926542989e−37	1.00918797088394	1.0	1.0	0.999999999999986	0.0621831695277462
8	−193.936309329378	21.2846727324735	1.0	1.0	0.999999988889254	0.105106868529532
16	261.855571189716	2.81210621846804	1.0	1.0	0.999987926041303	0.19375232975903
32	10.0185396377851	1.51015545835249	1.0	1.0	0.999560061848654	0.347596269917032
64	2.73116678859359	1.19563555686237	1.0	1.0	0.997842086626212	0.542932766367849
128	1.58813893510845	1.08427584722217	1.0	1.0	0.996539943503618	0.718650334044564
256	1.24691418359867	1.03828503418819	1.0	1.0	0.996762823873113	0.841498406774183
512	1.11336584348318	1.01787931107158	1.0	1.0	0.997648733023225	0.91528104994924
1024	1.05424085906049	1.00848394015116	1.0	1.0	0.998498801065311	0.956020252940866

Second order term

a \ z	−100	−10	−1	1	10	100
2	5.09546842383166e−81	1.0000344358472	1.0	1.0	1.0	0.56499453681103
4	3.50389347149145e−36	0.955827111745887	1.0	1.0	1.00000000000003	0.562183027622633
8	1310.75154218397	−24.4605562630437	1.0	1.0	1.00000000399844	0.605053287218248
16	−704.465792608461	0.327977035777927	1.0	1.0	1.00000095360871	0.690514097867283
32	−7.89953878007251	0.929712698541761	1.0	1.0	1.0000079956452	0.812622055181095
64	0.340363904011595	0.988687580859598	1.0	1.0	1.00000576108694	0.916242297492568
128	0.903723733597904	0.997827667122859	1.0	1.0	0.999996871461882	0.970510870252898
256	0.98138038593246	0.999545628736025	1.0	1.0	0.999995825820141	0.99106316295758
512	0.995911292221479	0.999900421319294	1.0	1.0	0.999997717520528	0.997512386453759
1024	0.999045993270933	0.999977552467753	1.0	1.0	0.999999075386754	0.999338848538859

$$a_{m,m}(z) = m \cdot a_{m-1,m-1}(z), \quad k = 0$$
$$\sum_{0\le j\le m-2} \binom{m}{j}(-1)^j \cdot a_{0,m-j}(z) = (-1)^m \cdot (m-1), \quad k = m.$$

This suggests the choice $a_{h,m}(z) = \frac{m!}{(m-h)!}$ for which (8) with $i = 0$ holds true. The final formula comes from the fact that $\sum_{h=0}^{n} \frac{n!}{(n-h)!a^h} = e^a a^{-n}\Gamma(n+1,a)$ (see, e.g., 8.8.2 in [33]). As concerns the second term, the recurrence (8) for $i = 1$ is:

$$\begin{aligned}\widetilde{\zeta}_1^{(m)}(z+1,a) &= -\sum_{j=1}^{m-2}\binom{m}{j}\left(\frac{j}{z}+\ln a\right)\cdot \ln^{j-1} a \cdot \widetilde{\zeta}_1^{(m-j)}(z+1,a) \\ &\quad + \left\{\frac{m(m-2)}{2z}\cdot \ln^{m-1} a + \frac{1}{2}(m-1)\ln^m a\right\}\cdot a^{-z-1}.\end{aligned}$$

This leads us to the choice $\widetilde{\zeta}_1^{(m)}(z+1,a) = \frac{(-1)^m}{2}\cdot \ln^m a \cdot a^{-z-1}$. As concerns $\widetilde{\Sigma}_m(z,a)$ in (7), we can write it as:

$$\begin{aligned}\widetilde{\Sigma}_m(z,a) - \Sigma_m(z,a) &= \sum_{j=1}^{m} c_{m,j}\cdot \widetilde{\Sigma}_{m-j}(z,a) \\ &= \sum_{j_0=1}^{m} c_{m,j_0}\Sigma_{m-j_0}(z,a) + \sum_{j_0=1}^{m-1}\sum_{j_1=1}^{m-j_0} c_{m,j_0}c_{m-j_0,j_1}\widetilde{\Sigma}_{m-j_0-j_1}(z,a) \\ &= \sum_{\ell=0}^{m-1}\sum_{j_0=1}^{m-\ell}\sum_{j_1=1}^{m-j_0}\cdots\sum_{j_\ell=1}^{m-j_0-\cdots-j_{\ell-1}} c_{m,j_0}\cdots c_{m-j_0-\cdots-j_{\ell-1},j_\ell}\Sigma_{m-j_0-\cdots-j_\ell}(z,a).\end{aligned}$$

Define $k_n := j_0 + \cdots + j_n$ for $0 \le n \le \ell$ and $k_{-1} := 0$. Then, for $n \ge 1$, $\sum_{j_n=1}^{m-j_0-\cdots-j_{n-1}} c_{m-j_0-\cdots-j_{n-1},j_n}$ can be written as $\sum_{k_{n-1}<k_n\le m} c_{m-k_{n-1},k_n-k_{n-1}}$ and

$$\begin{aligned}\widetilde{\Sigma}_m(z,a) &= \Sigma_m(z,a) \\ &\quad + \sum_{i=0}^{m-1}\Sigma_i(z,a)\cdot\left\{c_{m,m-i} + \sum_{\ell=1}^{m-i-1}\sum_{1\le k_0<\cdots<k_{\ell-1}<m-i}\left[\prod_{j=0}^{\ell-1} c_{m-k_{j-1},k_j-k_{j-1}}\right]c_{m-k_{\ell-1},m-i-k_{\ell-1}}\right\}.\end{aligned}$$

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