A Tight Bound on the Distance Between a Noncentral Chi Square and a Normal Distribution

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***Abstract*—We provide a nonasymptotic bound on the distance between a noncentral chi square distribution and a normal approximation. It improves on both the classical Berry-Esséen bound and previous distances derived specifically for this situation. First, the bound is nonasymptotic and provides an upper limit for the real distance. Second, the bound has the correct rate of decrease and even the correct leading constant when either the number of degrees of freedom or the noncentrality parameter (or both) diverge to infinity. The bound is applied to some probabilities arising in energy detection and Rician fading.**

***Index Terms*—Closed-form solutions, energy detection, probability, random variables, statistics, upper bound.**

I. Introduction

In this letter we address the problem, already considered in [8], of the accuracy of the approximation of a noncentral chi square random variable $\chi_k^2(s)$, with number of degrees of freedom k and noncentrality parameter s, through a normal one. The formula in [8], despite being very useful and displaying the correct rate of decrease, is an asymptotic description of the distance between the two distributions and does not provide an upper bound on it; this means that it is not guaranteed to majorize the true distance when k is finite (see Fig. 1 for an example in which the formula is smaller than the true distance). Moreover, it does not take into account the noncentrality parameter s. We improve the results in that paper in two directions. First, we provide nonasymptotic bounds valid for any $k \geq 1$. Second, we show that considering also the noncentrality parameter s, when available, yields generally smaller bounds and that a large noncentrality parameter improves the convergence rate. Our bound has the correct rate of decrease for either k or s (or both) diverging to infinity. Moreover, even the leading constant is exact, in the sense that for any combination of k and s with $k + s$ large, the ratio of our bound to the true value of the distance converges to 1.

II. Normal Approximations to $\chi_k^2(s)$

In the following we will explicitly consider a noncentral chi square random variable $\chi_k^2(s)$, while the central case $\chi_k^2 \equiv \chi_k^2(0)$ will arise taking $s = 0$. Now, $\chi_k^2(s)$ can be written as the sum of the squares of k independent normal random variables, X_i, with mean $\sqrt{s/k}$ and unit variance:

$$\chi_k^2(s) = \sum_{i=1}^{k} X_i^2. \tag{II.1}$$

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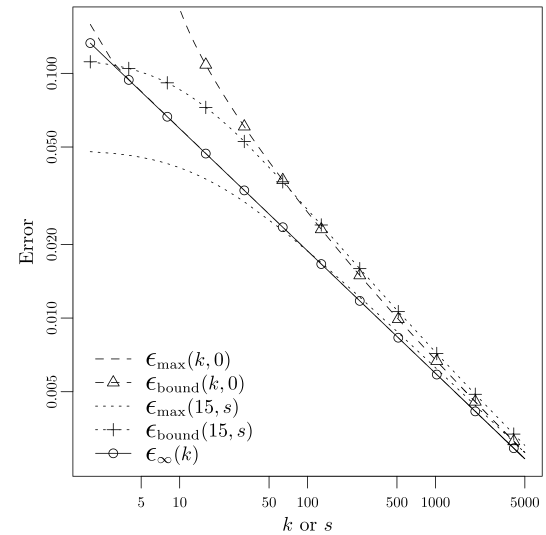

Fig. 1. Log-log plot of the error $\epsilon_{\max}(k, 0)$ and of its bounds $\epsilon_{\text{bound}}(k, 0)$ and $\epsilon_{\infty}(k)$, for k on the abscissa, and of the error $\epsilon_{\max}(15, s)$ and of its bound $\epsilon_{\text{bound}}(15, s)$, for s on the abscissa.

As a sum of k independent and identically distributed random variables, when k diverges the random variable $\chi_k^2(s)$ is subject to the Central Limit Theorem (CLT). Therefore a standardized version of $\chi_k^2(s)$ converges in distribution to a normal random variable when $k \to \infty$:

$$\frac{\chi_k^2(s) - (k+s)}{\sqrt{2k+4s}} \to_{\mathcal{D}} \mathcal{N}(0, 1). \tag{II.2}$$

Otherwise stated, the CLT says that, when $k \to \infty$:

$$\mathbb{P}\left\{\frac{\chi_k^2(s) - (k+s)}{\sqrt{2k+4s}} \leq x\right\} \to \mathbb{P}\left\{\mathcal{N}(0, 1) \leq x\right\}, \quad \forall x.$$

This convergence can be replaced by an approximate equality, even for k finite, in the forms:

$$\mathbb{P}\left\{\frac{\chi_k^2(s) - (k+s)}{\sqrt{2k+4s}} \leq x\right\} \approx \mathbb{P}\left\{\mathcal{N}(0, 1) \leq x\right\}, \tag{II.3}$$

and $\mathbb{P}\{\chi_k^2(s) \leq x\} \approx \mathbb{P}\{\mathcal{N}(k + s, \sqrt{2k + 4s}) \leq x\}$. This formula is proposed, e.g., in [10, Equation (29.67)]. It is quite similar to the approximation of $\chi_k^2(s)$ through $\mathcal{N}(k + s - 1, \sqrt{2k + 4s})$ in [9].

The limit theorem in (II.2) continues to hold true when k is fixed but $s \to \infty$. This less evident result is based on the fact that increasing $\sqrt{s/k}$ reduces the skewness of the generic summand X_i^2 in (II.1). This suggests that the approximation (II.3) remains true in this case too, a fact that we will show rigorously in the next section.

We define:

$$\epsilon(k, s, x) = \mathbb{P}\left\{\chi_k^2(s) \leq x\right\} - \mathbb{P}\left\{\mathcal{N}(k + s, \sqrt{2k + 4s}) \leq x\right\}$$

and $\epsilon_{\max}(k, s) = \sup_x |\epsilon(k, s, x)|$. In the next section we will provide an upper bound $\epsilon_{\text{bound}}(k, s)$ for $\epsilon_{\max}(k, s)$.

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III. Main Result

For $k \geq 8$, we have $\epsilon_{\max}(k, s) \leq \epsilon_{\text{bound}}(k, s)$ where:

$$\begin{aligned}\epsilon_{\text{bound}}(k, s) &= \frac{(k+4s)}{\pi W(1)(k+2s)^2}\left(1+\frac{k^2}{32}\binom{k}{8}^{-1/4}\right) \\ &\quad + \frac{(k+3s)}{3\sqrt{\pi}(k+2s)^{3/2}} \\ &\simeq 0.5612513 \cdot \frac{(k+4s)}{(k+2s)^2}\left(1+ 0.03125 \cdot k^2\binom{k}{8}^{-1/4}\right) \\ &\quad + 0.1880632 \cdot \frac{(k+3s)}{(k+2s)^{3/2}} \qquad \text{(III.1)}\end{aligned}$$

where $W(\cdot)$ is the Lambert W-function (see, e.g., [13, p. 111]) and $W(1) \simeq 0.5671433$. A proof of this result is contained in Appendix A. When $s \equiv 0$, we get the accuracy in the approximation of a chi square through a normal:

$$\epsilon_{\text{bound}}(k, 0) = \frac{1}{\pi W(1)k}\left(1+\frac{k^2}{32}\binom{k}{8}^{-1/4}\right) + \frac{1}{3\sqrt{\pi}k^{1/2}}.$$

As to the asymptotic bounds, we have $\epsilon_{\infty}(k) \leq k^{-1/2}/3\sqrt{\pi}$. As this is the same as the bound of [8], we can affirm that our asymptotic bound holds with equality. However, using (III.1) we can also provide an asymptotic bound depending on s:

$$\epsilon_{\infty}(k, s) \leq \frac{(k+3s)}{(k+2s)^{3/2}} \cdot \frac{1}{3\sqrt{\pi}} \leq \epsilon_{\infty}(k).$$

Remark 1:

(i) The bound (III.1) decreases in both k and s. Remark that for either $k \to \infty$ or $s \to \infty$, the limit of the noncentral chi square distribution is exactly the normal distribution, so that these bounds can be considered as accuracies in limit theorems, exactly as the Berry-Esséen bound.

(ii) When $s \geq 0$ is fixed and $k \to \infty$, the bound decreases as $0.1880632 \cdot k^{-1/2}$. This is the same rate of decrease of the Berry-Esséen Theorem (see, e.g., [16, p. 259]) and also of Theorem 1 in [8]. What is more remarkable is that even the leading constant is the same of Theorem 1 in [8] and, as the constant in their result, it is the best possible! Clearly, the presence of the first addendum slightly deteriorates the value of the bound, but asymptotically this addendum disappears.

(iii) When $k \geq 8$ is fixed and $s \to \infty$, the bound decreases as $0.1994711 \cdot s^{-1/2}$. This rate of decrease is tight, as remarked in [10, p. 466, after Equation (29.68)]. Moreover, in this case too, the leading constant is exact. We will show this below.

(iv) As an example, when $k = 250$, that is the value advocated by [17, p. 528], the bound is not larger than 0.01516183 (when $s = 0$), but it can be substantially better for large values of s. For the value proposed in [2, p. 118]], i.e., $k = 50$, the bound is not larger than 0.04358232. This value may seem too large. However, one should consider that in statistical applications the quality of the approximation in the right tail is what matters the most, and this is generally much better than the uniform distance between the two distributions.

(v) If $k > 8$, (III.1) can be simplified through the inequality $k^2\binom{k}{8}^{-1/4} \leq \frac{(8!)^{-1/4}}{(1-8/k)^2}$ (see [3, p. 158], [18, p. 283, Eq. (B.3)]).

(vi) A less elegant bound valid for any $k \geq 1$ is derived at the end of Appendix A:

$$\begin{aligned}\epsilon'_{\text{bound}}(k, s) &= \frac{k+4s}{\pi W(1)(k+2s)^2} + \frac{k+3s}{3\sqrt{\pi}(k+2s)^{3/2}} \\ &\quad + \frac{2(k+4s)^{(k\wedge 8)/8}}{(k\wedge 8)\pi\binom{k}{k\wedge 8}^{1/4}(8W(1))^{(k\wedge 8)/8}} \\ &\quad \cdot \exp\left\{-\frac{\sqrt{2W(1)}s}{\sqrt{k+4s}+\sqrt{8W(1)}}\right\}. \qquad \text{(III.2)}\end{aligned}$$

This equals $\epsilon_{\text{bound}}(k, 0)$ when $s = 0$ and is useful for large s.

When k and/or s are large, the noncentral chi square distribution approaches a normal one. Therefore, an Edgeworth expansion in the style of [1] can be obtained as:

$$\epsilon(k, s, x) = \frac{(1-x^2)e^{-\frac{x^2}{2}}}{3\sqrt{\pi}} \cdot \frac{(k+3s)}{(k+2s)^{3/2}} + o\left(\frac{(k+3s)}{(k+2s)^{3/2}}\right).$$

This implies that $\epsilon_{\infty}(k, s) \simeq \frac{1}{3\sqrt{\pi}} \cdot \frac{(k+3s)}{(k+2s)^{3/2}}$, so that our bound is asymptotically tight when k, s or both diverge to infinity. Fig. 1 displays $\epsilon_{\max}(k, 0)$, $\epsilon_{\text{bound}}(k, 0)$ and $\epsilon_{\infty}(k)$ as functions of k, and $\epsilon_{\max}(15, s)$ and $\epsilon_{\text{bound}}(15, s)$ as functions of s. The plot shows that $\epsilon_{\text{bound}}(k, 0)$ and $\epsilon_{\text{bound}}(15, s)$ are tight upper bounds respectively for $\epsilon_{\max}(k, 0)$ and $\epsilon_{\max}(15, s)$, and that $\epsilon_{\infty}(k)$ may be smaller than the true error $\epsilon_{\max}(k, 0)$.

IV. Application to Energy Detection

Consider the situation described in [7], [8], [17]. We have two hypotheses:

$$\begin{aligned}\mathcal{H}_0 &: r[n] = w[n] \qquad\qquad n = 1, 2, \ldots, N \\ \mathcal{H}_1 &: r[n] = s[n] + w[n] \quad n = 1, 2, \ldots, N\end{aligned}$$

where $r[n]$, $w[n]$ and $s[n]$ is the n-th observation of the received, noise only and trasmitted signal. N is the sample size, $\mathcal{H}_0$ is the null hypothesis, i.e., the channel is unoccupied, and $\mathcal{H}_1$ the alternative one, i.e., the channel is occupied, and all random variables composing the noise process are independent and normally distributed.

The test statistic computed by the energy detector is usually $T = \frac{1}{\sigma^2}\sum_{n=1}^{N}|r[n]|^2$, where σ^2 is the power of the noise signal. Under the null hypothesis $\mathcal{H}_0$, T is distributed according to a chi square random variable with N degrees of freedom. If γ denotes the signal to noise ratio, then, under the alternative hypothesis $\mathcal{H}_1$, T is distributed according to a noncentral chi square random variable with N degrees of freedom and noncentrality parameter γN.

If λ is the decision threshold, the probability of false alarm is $\mathbb{P}_f = \mathbb{P}\{T > \lambda|\mathcal{H}_0\} = \mathbb{P}\{\chi^2_N > \lambda\}$, while the probability of detection is $\mathbb{P}_d(\gamma) = \mathbb{P}\{T > \lambda|\mathcal{H}_1\} = \mathbb{P}\{\chi^2_N(\gamma N) > \lambda\}$. Both these probabilities can be approximated through the CLT as in (III.1).

As concerns the probability of detection $\mathbb{P}_d(\gamma)$, we have:

$$\epsilon_{\max}(N, \gamma N) \leq \frac{(1+4\gamma)N^{-1}}{\pi W(1)(1+2\gamma)^2}\left(1+\frac{N^2}{32}\binom{N}{8}^{-1/4}\right) + \frac{(1+3\gamma)N^{-1/2}}{3\sqrt{\pi}(1+2\gamma)^{3/2}}.$$

As a large value of γ improves convergence to normality, bound (11) in [8], obtained in the case $\gamma = 0$, is conservative:

$$\epsilon_{\infty}(N, \gamma N) = \frac{(1+3\gamma)N^{-1/2}}{3\sqrt{\pi}(1+2\gamma)^{3/2}} \leq \epsilon_{\infty}(N) = \frac{N^{-1/2}}{3\sqrt{\pi}}.$$

When both approximations are used in the context of a receiver operating characteristic (ROC) curve, through the definition $\epsilon_{ROC} = \sqrt{\epsilon^2(N, 0, \lambda) + \epsilon^2(N, \gamma N, \lambda)}$ we obtain:

$$\epsilon_{\infty,ROC} \leq \sqrt{\epsilon^2_{\infty}(N, 0) + \epsilon^2_{\infty}(N, \gamma N)} \leq \sqrt{\frac{2+12\gamma+21\gamma^2+8\gamma^3}{9\pi(1+2\gamma)^3 N}}.$$

The corresponding bound in [8] is $\sqrt{2/9\pi N}$. For $\gamma = 0$ the two bounds coincide, but when $\gamma \to \infty$ the expression in [8] overestimates ours by about 41.41%. When $\gamma = 1$ (5, 10), this percentage is already 12.06% (29.51%, 34.61%).

V. Application to Rician Fading

Let $Z_i \sim \mathcal{N}(m_i, \sigma^2)$, $i=1, 2$, be two independent normal random variables. The envelope $R=\sqrt{Z_1^2 + Z_2^2}$ of the complex random variable $Z_1 + \mathrm{i}Z_2$ is a Rician random variable $\mathrm{Rice}(\mu, \sigma)$ with probability density function [5, Section 5.3.5]:

$$f_R(x) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2+\mu^2}{2\sigma^2}\right) I_0\left(\frac{x\mu}{\sigma^2}\right)$$

where $\mu = \sqrt{m_1^2 + m_2^2}$ and I_0 is the modified Bessel function of the first kind of order 0 [13, p. 249]. This describes the fluctuations of the signal envelope when there is a dominant component, often corresponding to a line-of-sight path between the transmitter and the receiver. Let $P_\mathrm{d} = 2\sigma^2$ be the power of the diffuse multipath component and $V_1 = \mu$ the amplitude of the remaining dominant component. Then, $R \sim \mathrm{Rice}(V_1, \sqrt{P_\mathrm{d}/2})$ implies that $R^2 \sim \frac{P_\mathrm{d}}{2} \cdot \chi_2^2\left(2V_1^2/P_\mathrm{d}\right)$ and we can approximate it with a normal random variable through (III.2). If $K := V_1^2/P_\mathrm{d}$, the distance:

$$\sup_x \left| \mathbb{P}\{R \le x\} - \mathbb{P}\left\{\mathcal{N}(0,1) \le \frac{x^2 - P_\mathrm{d}(1+K)}{\sqrt{P_\mathrm{d}^2(1+2K)}}\right\}\right|$$

can be bounded by:

$$\begin{aligned}\epsilon'_{\mathrm{bound}}(k,s) =& \frac{1+4K}{2\pi W(1)(1+2K)^2} + \frac{1+3K}{3\sqrt{2\pi}(1+2K)^{3/2}} \\ &+ \frac{(1+4K)^{1/4}}{W(1)^{1/4}\pi 2^{1/2}} \cdot \exp\left\{-\frac{\sqrt{4W(1)}\cdot K}{\sqrt{1+4K}+\sqrt{4W(1)}}\right\} \\ \sim& \frac{1}{4\sqrt{\pi}K^{1/2}} \simeq 0.1410474 \cdot K^{-1/2}.\end{aligned}$$

In this case too it is apparent that the rate of convergence is optimal. This approximation can be linked, through the delta method [16, p. 279], to the one in [12, Appendix], [5, p. 126] or [14, p. 50]. It is also possible to show that both approximations have the same rate of convergence to 0.

The reasoning can be extended to the folded Gaussian distribution, when $k = 1$, or to the generalized Rician distribution [14, (2.3-64)].

VI. Application to Statistical Computations

The distribution function of $\chi_k^2(s)$ is customarily computed through series representations, that are truncated to achieve a given accuracy. When s is larger than 10^5, the number of iterations required for the computation may become too large and the produced result may be unreliable (see, e.g., the documentation of function `Chisquare` in [15]). As an example, for $x = s = 10^7$ and $k = 10$, R, Python and WolframAlpha yield respectively 0.1247757 (with a warning that the limit number of iterations has been reached), 0.49379561151252138 and 0.49943229536468405. The normal approximation provides the result 0.4993692 with an error bound equal to $6.320213 \cdot 10^{-5}$. This is reinforced by the fact that in many cases in which a probability of $\chi_k^2(s)$ with very large s is needed, one could actually use a different asymptotic approximation to get a normal limit (see, e.g., [4]).

VII. Conclusion

This letter elaborates on [8] and provides a nonasymptotic bound for the distance between a noncentral chi square distribution and its normal approximation. The bound is shown to be very precise when either k or s diverges to infinity.

It is fair to highlight two facts. On the one hand, despite the asymptotic bound in [8] is not guaranteed to provide an upper bound for the distance between the two distributions, in several cases it is larger than the true distance (but see Fig. 1 for an exception). On the other hand, the improvement deriving from s is not always large in practice (but see Section V for a case in which it is).

Appendix A
Proof of the Bound (III.1)

The proof uses the Fourier inversion formula to bound the distance between the two distributions through an integral involving the difference of the respective characteristic functions. The integral is then split in several parts corresponding to different regions of integration whose extremes depend on a parameter c, each part is majorized separately and the result is then optimized with respect to the parameter.

First of all, remark that:

$$\epsilon_{\max}(k,s) = \sup_x |\mathbb{P}\{X \le x\} - \mathbb{P}\{Y \le x\}|,$$

where $X := \frac{\chi_k^2(s)-(k+s)}{\sqrt{2(k+2s)}}$ and $Y := \mathcal{N}(0,1)$. From [6, p. 33]:

$$\sup_x |\mathbb{P}\{X \le x\} - \mathbb{P}\{Y \le x\}| \le \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left|\frac{\phi_X(t)-\phi_Y(t)}{t}\right| \mathrm{d}t$$

where $\phi_X := \mathbb{E}e^{\mathrm{i}tX}$ is the *characteristic function* of the random variable X. We recall that $\phi_Y(t) = e^{-t^2/2}$ and:

$$\begin{aligned}\phi_X(t) =& \mathbb{E}e^{\frac{\mathrm{i}t\chi_k^2(s)}{\sqrt{2(k+2s)}} - \frac{\mathrm{i}t(k+s)}{\sqrt{2(k+2s)}}} = \left(1 - \frac{2\mathrm{i}t}{\sqrt{2(k+2s)}}\right)^{-k/2} \\ &\cdot \exp\left(\frac{s\mathrm{i}t}{\sqrt{2(k+2s)}-2\mathrm{i}t} - \frac{\mathrm{i}t(k+s)}{\sqrt{2(k+2s)}}\right).\end{aligned}$$

As an intermediate step, we need the function:

$$\phi_{X^\star}(t) := \exp\left\{-\frac{t^2}{2} - \mathrm{i}\frac{2^{1/2}t^3}{3}\frac{(k+3s)}{(k+2s)^{3/2}}\right\}$$

that is not a proper characteristic function [11, p. 223, Theorem 7.3.5]. Then we have:

$$\begin{aligned}I :=& \int_{-\infty}^{+\infty}\left|\frac{\phi_X(t)-\phi_Y(t)}{t}\right|\mathrm{d}t \le \int_{|t|\le A}\left|\frac{\phi_X(t)-\phi_{X^\star}(t)}{t}\right|\mathrm{d}t \\ &+ \int_{|t|\le A}\left|\frac{\phi_{X^\star}(t)-\phi_Y(t)}{t}\right|\mathrm{d}t + \int_{|t|>A}\left|\frac{\phi_X(t)}{t}\right|\mathrm{d}t \\ &+ \int_{|t|>A}\left|\frac{\phi_Y(t)}{t}\right|\mathrm{d}t =: I_1 + I_2 + I_3 + I_4.\end{aligned}$$

1) We start from I_1. Define $\theta := -\frac{2t}{\sqrt{2(k+2s)}}$. Then we have (see [18, Eq. (B.1)]):

$$\begin{aligned}\phi_X(t) &= (1+\mathrm{i}\theta)^{-k/2}\exp\left(-\frac{s\mathrm{i}\theta}{2(1+\mathrm{i}\theta)} + \frac{\mathrm{i}\theta(k+s)}{2}\right) \\ \phi_{X^\star}(t) &= \exp\left\{-\frac{(k+2s)}{4}\theta^2 - \mathrm{i}\frac{(k+3s)}{6}\theta^3\right\}.\end{aligned}$$

From this:

$$I_1 = \int_{|t|\le A} \left|\frac{\phi_{X^\star}(t)}{t}\right| \left| e^{\ln \frac{\phi_X(t)}{\phi_{X^\star}(t)}} - 1 \right| \mathrm{d}t$$
$$\le \int_{|t|\le A} \left|\frac{\phi_{X^\star}(t)}{t}\right| \left|\ln \frac{\phi_X(t)}{\phi_{X^\star}(t)}\right| e^{\left|\ln \frac{\phi_X(t)}{\phi_{X^\star}(t)}\right|} \mathrm{d}t$$

where the last step comes from the inequality $|e^z - 1| \le |z|e^{|z|}$ (see, e.g., [16, p. 352, Eq. (3)]). Note that $|\phi_{X^\star}(t)| = \exp\left(-\frac{t^2}{2}\right)$. Now, following the development of [18, p. 282]:

$$\left|\ln \frac{\phi_X(t)}{\phi_{X^\star}(t)}\right| \le \frac{k}{2}\left|\ln(1+\mathrm{i}\theta) - \mathrm{i}\theta - \frac{\theta^2}{2} - \frac{\mathrm{i}\theta^3}{3}\right|$$
$$+ \frac{s|\theta|}{2}\left|(1+\mathrm{i}\theta)^{-1} - 1 + \mathrm{i}\theta - \theta^2\right|$$
$$\le \frac{k}{2}\frac{|\theta|^4}{4} + \frac{s|\theta|}{2}|\theta|^3 = t^4\frac{(k+4s)}{2(k+2s)^2}$$

where the second step comes from equations (B.2) in [18, p. 282]. Now we suppose that $t^4\frac{(k+4s)}{2(k+2s)^2} \le \frac{1}{c}$ for $|t| \le A$, where c will be chosen later. Thus, when $|t| = A$, we take $A^4\frac{(k+4s)}{2(k+2s)^2} = \frac{1}{c}$ or $A = \left\{c\frac{(k+4s)}{2(k+2s)^2}\right\}^{-1/4}$, and $\left|\ln \frac{\phi_X(t)}{\phi_{X^\star}(t)}\right| \le A^4\frac{(k+4s)}{2(k+2s)^2} = \frac{1}{c}$. Now:

$$I_1 \le \int_{|t|\le A} \left|\frac{\phi_{X^\star}(t)}{t}\right| \left|\ln \frac{\phi_X(t)}{\phi_{X^\star}(t)}\right| e^{\left|\ln \frac{\phi_X(t)}{\phi_{X^\star}(t)}\right|} \mathrm{d}t$$
$$\le \frac{(k+4s)}{2(k+2s)^2} e^{1/c} \int_{|t|\le A} e^{-t^2/2}|t|^3 \mathrm{d}t.$$

2) Now we pass to I_2. The technique is quite similar:

$$I_2 = \int_{|t|\le A} \left|\frac{\phi_Y(t)}{t}\right| \left| e^{\ln \frac{\phi_{X^\star}(t)}{\phi_Y(t)}} - 1 \right| \mathrm{d}t.$$

As $|\phi_Y(t)| = e^{-t^2/2}$ and $\ln \frac{\phi_{X^\star}(t)}{\phi_Y(t)} = -\mathrm{i}\frac{2^{1/2}t^3}{3}\frac{(k+3s)}{(k+2s)^{3/2}}$, from $|e^{\mathrm{i}x} - 1| \le |x|$ [16, p. 352, Eq. (4)]:

$$I_2 \le \frac{2^{1/2}(k+3s)}{3(k+2s)^{3/2}} \int_{|t|\le A} e^{-t^2/2} t^2 \mathrm{d}t$$
$$\le \frac{2^{1/2}(k+3s)}{3(k+2s)^{3/2}} \int_{-\infty}^{+\infty} e^{-t^2/2} t^2 \mathrm{d}t = \frac{2\sqrt{\pi}(k+3s)}{3(k+2s)^{3/2}}.$$

3) As concerns I_3, from [18, p. 283, Eq. (B.3)], $|\phi_X(t)|^{-4} \ge \left(1 + \frac{2t^2}{k}\right)^k = \sum_{j=0}^{k}\binom{k}{j}\left(\frac{2t^2}{k}\right)^j$. If $k \ge 8$, $|\phi_X(t)|^{-4} \ge \binom{k}{8}\left(\frac{2t^2}{k}\right)^8$ and:

$$I_3 \le \int_{|t|>A} \left|\frac{\phi_X(t)}{t}\right| \mathrm{d}t \le \frac{k^2}{8}\binom{k}{8}^{-1/4} A^{-4}.$$

4) As concerns I_4:

$$I_4 = \int_{|t|>A} \frac{e^{-t^2/2}}{|t|}\mathrm{d}t \le A^{-4}\int_{|t|>A} e^{-t^2/2}|t|^3 \mathrm{d}t.$$

Collecting all the terms, we get:

$$I \le \frac{(k+4s)}{2(k+2s)^2} e^{1/c} \int_{|t|\le A} e^{-t^2/2}|t|^3 \mathrm{d}t + \frac{2\sqrt{\pi}(k+3s)}{3(k+2s)^{3/2}}$$
$$+ \frac{k^2}{8}\binom{k}{8}^{-1/4} A^{-4} + A^{-4}\int_{|t|>A} e^{-t^2/2}|t|^3 \mathrm{d}t$$

and using $A^{-4} = c\frac{(k+4s)}{2(k+2s)^2}$, we get:

$$I \le \frac{(k+4s)}{(k+2s)^2}\left\{e^{1/c} \vee c\right\}\int_0^{+\infty} e^{-t^2/2}t^3 \mathrm{d}t$$
$$+ \frac{2\sqrt{\pi}(k+3s)}{3(k+2s)^{3/2}} + \frac{k^2(k+4s)}{16(k+2s)^2} c \binom{k}{8}^{-1/4}.$$

As this holds for any c, we choose c minimizing the first term, i.e., such that $e^{1/c} = c$ or $c = 1/W(1)$ where W is the Lambert W-function [13, p. 111]. From $\int_0^{+\infty} e^{-t^2/2}t^3 \mathrm{d}t = 2$:

$$I \le \frac{2(k+4s)}{W(1)(k+2s)^2}\left(1 + \frac{k^2}{32}\binom{k}{8}^{-1/4}\right) + \frac{2\sqrt{\pi}(k+3s)}{3(k+2s)^{3/2}}$$

and the final result follows by substitution.

To remove the condition $k \ge 8$, we only change I_3, majorizing separately the two terms of $|\phi_X(t)| = \left(1 + \frac{2t^2}{k+2s}\right)^{-k/4}\exp\left\{-\frac{st^2}{k+2s+2t^2}\right\}$ on the domain of integration:

$$\left(1 + \frac{2t^2}{k+2s}\right)^k = \sum_{j=0}^{k}\binom{k}{j}\left(\frac{2t^2}{k+2s}\right)^j \ge \binom{k}{k \wedge 8}\left(\frac{2t^2}{k+2s}\right)^{k\wedge 8},$$

$$\exp\left\{-\frac{st^2}{k+2s+2t^2}\right\} \le \exp\left\{-\frac{sA^2}{k+2s+2A^2}\right\}.$$

The result follows by integration and substitution.

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